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THE GENERALISED DINAMICAL PROBLEM OF THERMOELASTICTY FOR THE HOLLOW CYLINDER

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In this paper, a dynamical problem for an infinite hollow cylinder in the generalized theory of thermoelasticity has been considered. The precise solution is obtained by the use of the Laplace transform on time and the finite integral transform on space.

The point of departure is the system of one-dimensional equations [1]

$$\Delta T-\frac{1}{a_T}\frac{\partial T}{\partial t}-\eta\frac{\partial}{\partial t}(\Delta\Psi)=0 \qquad (1)$$

$$\Delta\Psi-\frac{1}{c_1^2}\frac{\partial^2\Psi}{\partial t^2}=mT \qquad (2)$$

subject to the following initial and boundary conditions

$$T=\Psi=\frac{\partial\Psi}{\partial t}=0,\ \text{if}\ \ t=0\,; \qquad (3)$$

$$\frac{\partial T}{\partial r}=g_j(t),\ \frac{\partial\Psi}{\partial r}=h_j(t),\ \text{if}\ r=R_j(j=1,2), \qquad (4)$$

where $\Delta=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}$; $\Psi(r,t)$ is the thermoelastic potential, defined by the formula $u(r,t)=\frac{\partial\Psi}{\partial r}$; $u(r,t)$ is the displacement, a_T is the thermal diffusivity, $\eta=\frac{2G(1+\nu)}{(1-2\nu)\lambda_T}\alpha_T T_0$ is the coupling constant, α_T is the coefficient of linear thermal expansion, $c_1=\sqrt{\frac{2G}{(1-2\nu)\rho}}$, G is the modulus of elasticity, ν is the Poisson's ratio, ρ is the density, $m = \frac{1+\nu}{1-\nu}\alpha_T$, λ_T is the coefficient of thermal conductivity.

The solution of the problem (1)-(4) can be thought of in the form

$$T(r, t) = T^*(r, t) + \varphi(r, t),$$
$$\Psi(r, t) = \Psi(r, t) + \psi(r, t), \tag{5}$$

where

$$\varphi(r, t) = A(t)\ln r + B(t)r^2, \quad \psi(r, t) = C(t)\ln r + D(t)r^2,$$

$$A(t) = \frac{2}{\Delta}(R_2 g_1(t) - R_1 g_2(t)), \quad B(t) = \frac{1}{\Delta}(g_2(t)/R_1 - g_1(t)/R_2),$$

$$C(t) = \frac{2}{\Delta}(R_2 h_1(t) - R_1 h_2(t)), \quad D(t) = \frac{1}{\Delta}(h_2(t)/R_1 - h_1(t)/R_2),$$

$$\Delta = 2(R_2^2 - R_1^2)/R_1 R_2 .$$

The substitution of (5) into (1)-(4) yields

$$\Delta T^* - \frac{1}{a_T}\frac{\partial T^*}{\partial t} = f_1(r,t), \quad \Delta\Psi^* - \frac{1}{c_1^2}\frac{\partial^2\Psi}{\partial t} = mT^* + f_2(r,t), \tag{6}$$

$$T^*(r,0) = -\varphi(r,0), \quad \Psi^*(r,0) = -\psi(r,0), \quad \frac{\partial\Psi^*(r,0)}{\partial t} = -\frac{\partial\psi(r,0)}{\partial t},$$

$$\frac{\partial T^*}{\partial r} = \frac{\partial\Psi^*}{\partial r} = 0 \text{ if } r = R_j, \tag{7}$$

where $f_1(r,t) = 4\eta\frac{dD(t)}{dt} - A(t) - \frac{1}{a_t}\frac{\partial\varphi}{\partial t}$,

$$f_2(r,t) = m\varphi(r,t) + \frac{1}{c_1^2}\frac{\partial^2\psi}{\partial t^2} - 4D(t).$$

We solve the problem (6), (7) by using the finite integral transforms [2].

To do this we find the solution of the following Sturm-Liouville problem [3]

$$\frac{d^2W}{dr^2} + \frac{1}{r}\frac{dW}{dr} + \gamma^2 W = 0,$$
$$W'(r) = 0 \text{ if } r = R_j (j = 1,2) \tag{8}$$

We obtain

$$W_n(r) = A_n J_0(\gamma_n r) + B_n Y_0(\gamma_n r),$$

$$A_n = \frac{Y_0(\gamma_n R_1) - R_1 Y'_0(\gamma_n R_1)}{J_0(\gamma_n R_1) - R_1 J'_0(\gamma_n R_1)} B_n,$$

where $J_0(\gamma_n R_1)$ and $Y_0(\gamma_n R_1)$ are the Bessel functions of the first and the second kind of order zero.

The coefficient B_n is defined from the condition

$$\int_{R_1}^{R_2} W_n^2(r) r\,dr = 1.$$

The eigenvalues γ_n satisfy the equation

$$J'_0(\gamma_n R_1) Y'_0(\gamma_n R_2) - J'_0(\gamma_n R_2) Y'_0(\gamma_n R_1) = 0. \tag{9}$$

The finite transforms and the inversion formulas are

$$T_n^*(t) = \int_{R_1}^{R_2} T^*(r,t) W_n(r) r\,dr, \quad \Psi_n^*(t) = \int_{R_1}^{R_2} \Psi^*(r,t) W_n(r) r\,dr \tag{10}$$

$$T^*(r,t) = \sum_{n=1}^{\infty} T_n^*(t) W_n(r), \quad \Psi_n^*(r,t) = \sum_{n=1}^{\infty} \Psi_n^*(t) W_n(r) \tag{11}$$

Applying the transforms (10) to the system of equations (6) and to the boundary conditions (7) we arrive at the following system of ordinary differential equations with respect to $T_n^*(t)$ and $\Psi_n^*(t)$

$$\begin{aligned} &\frac{1}{a_T}\frac{dT_n^*}{dt} - \eta\gamma_n^2 \frac{d\Psi_n^*}{dt} + \gamma_n^2 T_n^* = f_{1n}(t), \\ &\frac{1}{c_1^2}\frac{d^2\Psi_n^*}{dt^2} + \gamma_n^2 \Psi_n^* + mT_n^* = f_{2n}(t) \end{aligned} \tag{12}$$

with the initial conditions

$$T_n^*(0) = -\varphi_n(0), \quad \Psi_n^*(0) = -\psi_n(0), \quad \frac{d\Psi_n^*}{dt} = -\frac{d\psi_n(0)}{dt} \tag{13}$$

Employing the Laplace transform to the system of the equations (12), (13) and solving it we get

$$T_n^*(t) = \sum_{k=0}^{2} A_{kn} F_{kn}(t) + \int_0^t f_{1n}(\tau)\left[a_T F_{2n}(t-\tau) + \right.$$

$$\left. + a_T c_1^2 \gamma_n^2 F_{0n}(t-\tau)\right] d\tau + \eta a_T c_1^2 \gamma_n^2 \int_0^t f_{2n}(\tau) F_{1n}(t-\tau) d\tau,$$

$$\Psi_n^*(t)=\sum_{k=0}^{2} B_{kn}F_{kn}(t)+c_1^2\int_0^t f_{2n}(\tau)F_{1n}(t-\tau)d\tau+$$
$$+a_T c_1^2\int_0^t\left[\gamma_n^2 f_{2n}(\tau)-m f_{1n}(\tau)\right]F_{0n}(t-\tau)d\tau, \quad (14)$$

where $F_{0n}(t)=\dfrac{e^{-a_n t}}{\Delta_{1n}}+\dfrac{e^{-b_n t}}{\Delta_{2n}}\sin(c_n t+\varphi_n)$,

$$F_{1n}(t)=-\frac{a_n e^{-a_n t}}{\Delta_{1n}}+\frac{e^{-b_n t}\rho_n}{\Delta_{2n}}\left[c_n\sin(c_n t+\varphi_{2n})-b_n\sin(c_n t+\varphi_{1n})\right],$$

$$F_{2n}(t)=-\frac{a_n^2 e^{-a_n t}}{\Delta_{1n}}-\frac{e^{-b_n t}\rho_n}{\Delta_{2n}}\left[\left(c_n^2-b_n^2\right)\sin(c_n t+\varphi_{1n})+\right.$$
$$\left.+2b_n c_n\sin(c_n t+\varphi_{2n})\right],$$

$$\Delta_{1n}=c_n^2+(b_n-a_n)^2,\quad \Delta_{2n}=c_n^4+(a_n c_n-b_n c_n)^2,$$

$$A_{0n}=a_T c_1^2\left[\eta\gamma_n^4\psi_n(0)-\frac{\gamma_n^2}{a_T}\varphi_n(0)\right],\ A_{1n}=-\eta a_T\gamma_n^2\psi_n(0),$$

$$A_{2n}=a_T\eta\gamma_n^2\psi_n(0)-\varphi_n(0)-a_T\eta\gamma_n^2\frac{d\psi_n(0)}{dt},$$

$$B_{0n}=-mc_1^2\varphi_n(0)-a_T\gamma_n^2\psi_n(0)\left(m\eta c_1^2+1\right),$$

$$B_{1n}=-a_T\gamma_n^2\frac{d\psi_n(0)}{dt}+\psi_n(0),\ B_{2n}=-\frac{d\psi_n(0)}{dt}, \quad (15)$$

$$\rho_n=\sqrt{\Delta_{2n}},\ \mathrm{tg}\varphi_{1n}=\frac{c_n}{b_n-a_n},\ \mathrm{tg}\varphi_{2n}=\frac{a_n-b_n}{c_n}$$

and $p_1=-a_n$, $p_{23}=-b_n\pm ic_n$ $(a_n>0,\ b_n>0)$ are the roots of the equation

$$p^3+a_T\gamma_n^2p^2+c_1^2\gamma_n^2(1-a_T m\eta)p+a_T c_1^2\gamma_n^4=0 \quad (16)$$

Finally the solution of the problem (1)-(4) can be written in the form

$$T(r,t)=\varphi(r,t)+\sum_{n=1}^{\infty}T_n^*(t)W_n(r),\ u(r,t)=\frac{\partial\psi(r,t)}{\partial r}+$$
$$+\sum_{n=1}^{\infty}\Psi_n^*(t)W_n'(r),\quad (t\ge 0,\ R_1\le r\le R_2). \quad (17)$$

Next, we consider the special case, when the harmonic displacement is given at the surface of the cylinder and the heat flow equals zero, that is

$$\frac{\partial T}{\partial r}=0,\ u=\frac{\partial \Psi}{\partial r}=u_j e^{-i\omega t}\ \text{if}\ r=R_j. \tag{18}$$

The solution of the problem (12) leads to the relations

$$T_n^*(t)=\frac{1}{\delta_n}\left[e_{1n}\left(\gamma_n^2-\frac{\omega^2}{c_1^2}\right)-m\eta\gamma_n^2 e_{2n}\right]e^{-i\omega t}, \tag{19}$$

$$\Psi_n^*(t)=\frac{1}{\delta_n}\left[e_{2n}\left(\gamma_n^2-\frac{i\omega}{a_T}\right)-me_{1n}\right]e^{-i\omega t}, \tag{20}$$

where $e_{1n}=\frac{4\eta}{\Delta}\left(\frac{u_2}{R_1}-\frac{u_1}{R_2}\right)$, $e_{2n}=\frac{1}{c_1^2\Delta}\left[2(R_2u_1-R_1u_2)\int_{R_1}^{R_2}\ln rW_n(r)r\,dr-\right.$

$$\left.-\left(\frac{u_2}{R_1}-\frac{u_1}{R_2}\right)\int_{R_1}^{R_2}r^3W_n(r)dr\right],$$

$$\delta_n=\left(\gamma_n^2-\frac{\omega^2}{c_1^2}\right)\left(\gamma_n^2-\frac{i\omega}{a_T}\right)-m\eta\gamma_n^2.$$

It is clear from (19) that $T_n^*=0$ if the coupling constant $\eta=0$ and hence $T_n(t)=0$ since $\varphi(t)=0$. The temperature disturbances can be large if the value $\eta\neq 0$ is however small and the wave number $\frac{\omega}{c_1}$ is close to γ_n.

Indeed, let $\frac{\omega}{c_1}=\gamma_n$ (resonance conditions), then we get from the formula (19)

$$T_1^*(t)=e_{21}e^{-i\omega t} \tag{21}$$

and the amplitude of these vibrations (not depending on η) can reach however large values.

References

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