Acta Mech 230, 871–883 (2019)
https://doi.org/10.1007/s00707-018-2328-7

ORIGINAL PAPER

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On a new type of solving procedure for Euler–Poisson equations (rigid body rotation over the fixed point)

Received: 27 May 2018 / Published online: 11 December 2018

Abstract In this paper, we proceed to develop a new approach which was formulated first in Ershkov (Acta Mech 228(7):2719–2723, 2017) for solving Poisson equations: a new type of the solving procedure for Euler–Poisson equations (rigid body rotation over the fixed point) is suggested in the current research. Meanwhile, the Euler–Poisson system of equations has been successfully explored for the existence of analytical solutions. As the main result, a new ansatz is suggested for solving Euler–Poisson equations: the Euler–Poisson equations are reduced to a system of three nonlinear ordinary differential equations of first order in regard to three functions Ω_i ($i = 1, 2, 3$); the proper elegant approximate solution has been obtained as a set of quasi-periodic cycles via re-inversing the proper elliptical integral. So the system of Euler–Poisson equations is proved to have analytical solutions (in quadratures) only in classical simplifying cases: (1) Lagrange's case, or (2) Kovalevskaya's case or (3) Euler's case or other well-known but particular cases.

1 Introduction, equations of motion

Euler–Poisson equations, describing the dynamics of rigid body rotation, are known to be one of the famous problems in classical mechanics.

In accordance with [1–3], Euler equations describe the rotation of a rigid body in a frame of reference fixed in the rotating body for the case of rotation over the fixed point as below (at given initial conditions):

$$\begin{cases} I_1 \frac{\mathrm{d}\Omega_1}{\mathrm{d}t} + (I_3 - I_2) \cdot \Omega_2 \cdot \Omega_3 = P\,(\gamma_2 c_0 - \gamma_3 b_0)\,, \\ I_2 \frac{\mathrm{d}\Omega_2}{\mathrm{d}t} + (I_1 - I_3) \cdot \Omega_3 \cdot \Omega_1 = P\,(\gamma_3 a_0 - \gamma_1 c_0)\,, \\ I_3 \frac{\mathrm{d}\Omega_3}{\mathrm{d}t} + (I_2 - I_1) \cdot \Omega_1 \cdot \Omega_2 = P\,(\gamma_1 b_0 - \gamma_2 a_0)\,, \end{cases} \tag{1.1}$$

where $I_i \neq 0$ are the principal moments of inertia ($i = 1, 2, 3$) and Ω_i are the components of the angular velocity vector along the proper principal axis; γ_i are the components of the weight of mass P and a_0, b_0, c_0 are the appropriate coordinates of the center of masses in a frame of reference fixed in the rotating body (in regard to the absolute system of coordinates X, Y, Z).

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The Poisson equations for the components of the weight in a frame of reference fixed in the rotating body (in regard to the absolute system of coordinates X, Y, Z) can be presented as below [4,5]:

$$\begin{cases} \frac{d\gamma_1}{dt} = \Omega_3\gamma_2 - \Omega_2\gamma_3, \\ \frac{d\gamma_2}{dt} = \Omega_1\gamma_3 - \Omega_3\gamma_1, \\ \frac{d\gamma_3}{dt} = \Omega_2\gamma_1 - \Omega_1\gamma_2, \end{cases} \tag{1.2}$$

besides, we should present the invariants (first integrals of motion) as

$$\begin{cases} \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1, \\ I_1 \cdot \Omega_1 \cdot \gamma_1 + I_2 \cdot \Omega_2 \cdot \gamma_2 + I_3 \cdot \Omega_3 \cdot \gamma_3 = \text{const.} = C_0, \\ \frac{1}{2}\left(I_1 \cdot \Omega_1^2 + I_2 \cdot \Omega_2^2 + I_3 \cdot \Omega_3^2\right) + P(a_0\gamma_1 + b_0\gamma_2 + c_0\gamma_3) = \text{const.} = C_1. \end{cases} \tag{1.3}$$

2 Derivation of the invariants (first integrals) of motion

According to [6], let us recall how to derive the invariants (1.3). From (1.1), (1.2) we obtain

$$\begin{aligned} &\frac{\frac{d}{dt}\left(I_1\frac{\Omega_1^2}{2} + I_2\frac{\Omega_2^2}{2} + I_3\frac{\Omega_3^2}{2}\right)}{P} = \Omega_1 \cdot (\gamma_2 c_0 - \gamma_3 b_0) + \Omega_2 \cdot (\gamma_3 a_0 - \gamma_1 c_0) + \Omega_3 \cdot (\gamma_1 b_0 - \gamma_2 a_0)\,, \\ \Rightarrow\quad &-\frac{\frac{d}{dt}\left(I_1\frac{\Omega_1^2}{2} + I_2\frac{\Omega_2^2}{2} + I_3\frac{\Omega_3^2}{2}\right)}{P} = a_0 \cdot (\Omega_3\gamma_2 - \Omega_2\gamma_3) + b_0 \cdot (\Omega_1\gamma_3 - \Omega_3\gamma_1) + c_0 \cdot (\Omega_2\gamma_1 - \Omega_1\gamma_2)\,, \\ \Rightarrow\quad &\frac{d}{dt}\left(I_1\frac{\Omega_1^2}{2} + I_2\frac{\Omega_2^2}{2} + I_3\frac{\Omega_3^2}{2}\right) + a_0 \cdot P \cdot \frac{d\gamma_1}{dt} + b_0 \cdot P \cdot \frac{d\gamma_2}{dt} + c_0 \cdot P \cdot \frac{d\gamma_3}{dt} = 0. \end{aligned}$$

So we have obtained the third integral of (1.3) in [7]. For obtaining the second integral of (1.3) in [7], we multiply each of Eqs. (1.1) by γ_i accordingly, but also each of equations of (1.2) by $(I_i \cdot \Omega_i)$ accordingly; then we add as below:

$$\begin{cases} \left(I_1 \cdot \gamma_1 \cdot \frac{d\Omega_1}{dt} + \gamma_1 \cdot (I_3 - I_2) \cdot \Omega_2 \cdot \Omega_3\right) + \left(I_1 \cdot \Omega_1 \frac{d\gamma_1}{dt}\right) = \gamma_1 \cdot P\,(\gamma_2 c_0 - \gamma_3 b_0) + I_1 \cdot \Omega_1 \cdot (\Omega_3\gamma_2 - \Omega_2\gamma_3), \\ \left(I_2 \cdot \gamma_2 \cdot \frac{d\Omega_2}{dt} + \gamma_2 \cdot (I_1 - I_3) \cdot \Omega_3 \cdot \Omega_1\right) + \left(I_2 \cdot \Omega_2 \frac{d\gamma_2}{dt}\right) = \gamma_2 \cdot P\,(\gamma_3 a_0 - \gamma_1 c_0) + I_2 \cdot \Omega_2 \cdot (\Omega_1\gamma_3 - \Omega_3\gamma_1), \\ \left(I_3 \cdot \gamma_3 \cdot \frac{d\Omega_3}{dt} + \gamma_3 \cdot (I_2 - I_1) \cdot \Omega_1 \cdot \Omega_2\right) + \left(I_3 \cdot \Omega_3 \frac{d\gamma_3}{dt}\right) = \gamma_3 \cdot P\,(\gamma_1 b_0 - \gamma_2 a_0) + I_3 \cdot \Omega_3 \cdot (\Omega_2\gamma_1 - \Omega_1\gamma_2), \end{cases}$$

Having done this, we add all the three equations above:

$$I_1 \cdot \frac{d}{dt}(\Omega_1 \cdot \gamma_1) + I_2 \cdot \frac{d}{dt}(\Omega_2 \cdot \gamma_2) + I_3 \cdot \frac{d}{dt}(\Omega_3 \cdot \gamma_3) = 0.$$

So we have obtained the second integral of (1.3) in [7].

The first integral of (1.3) is trivial, but belongs to Poisson equations only: to obtain it, we multiply each of equations of (1.2) on γ_i accordingly, then add them (the constant of integration is chosen equal to 1, due to trigonometric sense of the presenting solution in absolute system of coordinates via Euler angles):

$$\frac{1}{2}\frac{d}{dt}(\gamma_1^2) + \frac{1}{2}\frac{d}{dt}(\gamma_2^2) + \frac{1}{2}\frac{d}{dt}(\gamma_3^2) = 0,$$

As we can see, 2 of 3 proper additional invariants above are obtained by using of all the 6 EP-equations (including Poisson equations).

But aforesaid argument is not sufficient for solving the EP-equations: indeed, the system of equations (1.1)–(1.2) is supposed to not be equivalent to the system of equations (1.1) along with all the invariants (1.3) (Dr. Hamad H. Yehya, personal communications) for some particular cases, as was suggested earlier in [7,8]. The rather complex case, which describes the motion of the constrained rigid body around a fixed point, was considered in the comprehensive article [9].

So, for solving the system of equations (1.1)–(1.2), we first solve the Poisson equations (1.2).