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MATHEMATICS IN ENGINEERING, SCIENCE AND AEROSPACE
MESA - www.journalmesa.com
Vol. 6, No. 3, pp. 383-391, 2015

Rotational motion of a satellite with viscous fluid under the action of the external resistance torque

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Abstract. Rapid rotational motion of a dynamically asymmetric satellite relative to the center of mass is studied. The satellite has a cavity filled with viscous fluid at low Reynolds numbers, and it moves under the action of gravitational torque and the external resistance torque. The rotational motions are considered within of the model of a quasi-rigid body whose center of mass moves in a circular orbit around the Earth. The problems of dynamics, generalized and complicated by accounting for various disturbing factor remain rather topical till now.

1 Introduction

Consider the motion of a rigid body or satellite about the center of mass under the action of the external resistance torque in the gravitational field. The body contains a cavity fully filled with highly viscous homogeneous fluid. Interest in problems in the dynamics of bodies with cavities containing fluid has grown considerably in connection with rapid development of missile and space technology. The fluid fuel tanks on board a rocket, satellite or spacecraft can significantly influence the motion of these flight vehicles. Rotations are considered within the model of a quasi-rigid body, whose center of mass moves in a circular orbit around the Earth. Such problem investigated in[1, 2, 3, 4, 5]. The problems of dynamics, generalized and complicated by accounting for various disturbing factors, remain rather topical till now. The papers [6, 7, 8, 9, 10, 11, 12, 13, 14, 15] study rotational motions of bodies about a center of mass under the action of perturbation torques of various nature (gravitational, influence of a cavity filled with viscous fluid, resistant medium, etc). Papers are concerned with the motion of a rigid body having a cavity that is completely filled with a viscous fluid. The basic assumption is that the Reynolds number is small. In papers [1, 5, 9, 10] satellite motion under the action of gravitational torques was studied. In article [7] fast rotational motion of a heavy rigid body about a fixed point when external resistance is present was investigated. Fast rotation of a dynamically asymmetric

2010 **70E05, 70E15, 70F15, 34C29, 76A02**
Keywords: Satellite, gravitational field, resistance torque, averaging method

satellite about the center of mass under the action of the gravitational torque and the drag torque was studied in [9]. Secular changes in the rotational motion of a planet due to dissipation of energy in its core are investigated in [11]. Optimal control of motion of a rigid body about its center of mass had received much attention in [16, 17, 18]. Optimal stabilization problem and deceleration of a rigid body considered under various assumptions about the dynamic characteristics of these bodies, control systems and for different performance criterion.The problem of time-optimal deceleration of rotation of a free rigid body is studied in [16]. It is assumed that the body contains a spherical cavity filled with highly viscous liquid. Low decelerating moment of viscous friction forces also acts on the rigid body. It is assumed that the body is dynamically symmetric or asymmetric. An optimal control law for the deceleration of rotation of the body is synthesized, and the corresponding time and phase trajectories are determined. In the monograph [17] examines the problem of the dynamics, control and optimization of mechanical systems with regard for elasticity structures. The problem of axisymmetric rigid body angular velocity vector maneuvering in the body-fixed frame is considered in [18].

2 Statement of the problem

We introduce three Cartesian coordinate systems whose origins coincide with the satellite center of inertia [1, 5]. The coordinate system Ox_i $(i = 1,2,3)$ moves translationally together with the center of inertia: the axis Ox_1 is parallel to the position vector of the orbit perigee, the axis Ox_2 is parallel to the velocity vector of the satellite center of mass at the perigee, and the axis Ox_3 is parallel to the normal to the orbit plane. The coordinate system Oy_i $(i = 1,2,3)$ is attached to the satellite and oriented along the angular momentum vector $\mathbf{G}$. The axis Oy_3 is directed along $\mathbf{G}$, the axis Oy_2 lies in the orbit plane Ox_1x_2, and the axis Oy_1 lies in the plane Ox_3y_3 and is directed so that the vectors $\mathbf{y}_1$,$\mathbf{y}_2$,$\mathbf{y}_3$ form a right trihedral [1, 5]. The axes of the coordinate system Oz_i $(i = 1,2,3)$ are related to the principal central axes of inertia of the rigid body. The mutual position of the principal central axes of inertia and the axes Oy_i is determined by the Euler angles. The direction cosines α_{ij} of the axes $0z_i$ with respect to the system Oy_i are expressed via the Euler angles φ,θ,ψ by well-known formulas [1]. The position of the angular momentum vector $\mathbf{G}$ with respect to the center of mass in the coordinate system Ox_i is determined by the angles λ and δ as shown in Fig. 1. The equations of motion of the body about the center of mass are written in the form [5]

$$\frac{dG}{dt} = L_3, \frac{d\delta}{dt} = \frac{L_1}{G}, \frac{d\lambda}{dt} = \frac{L_2}{G\sin\delta}$$

$$\frac{d\theta}{dt} = G\sin\theta\sin\varphi\cos\varphi\left(\frac{1}{A_1}-\frac{1}{A_2}\right)+\frac{L_2\cos\psi-L_1\sin\psi}{G}$$

$$\frac{d\varphi}{dt} = G\cos\theta\left(\frac{1}{A_3}-\frac{\sin^2\varphi}{A_1}-\frac{\cos^2\varphi}{A_2}\right)+\frac{L_1\cos\psi+L_2\sin\psi}{G\sin\theta} \tag{2.1}$$

$$\frac{d\psi}{dt} = G\left(\frac{\sin^2\varphi}{A_1}+\frac{\cos^2\varphi}{A_2}\right)-\frac{L_1\cos\psi+L_2\sin\psi}{G}ctg\theta-\frac{L_2}{G}ctg\delta$$

Here, L_i are the torques of applied forces about the axes Oy_i, G is the value of the angular momentum, and the A_i $(i = 1,2,3)$ are the principal central moments of inertia about the axes Oz_i. In some cases, along with the variable θ, it is convenient to use an important characteristic, the kinetic energy T as an additional variable. It‘s derivative has the form

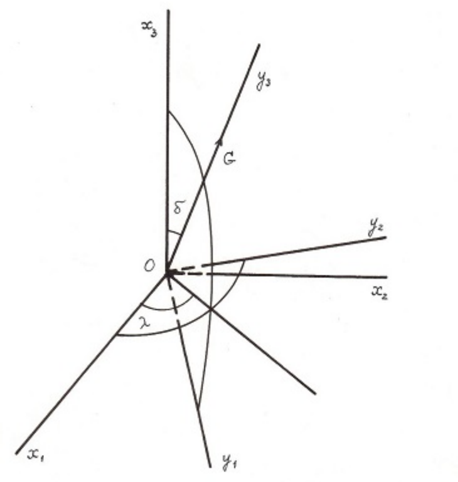

Fig. 1

$$\frac{dT}{dt} = \frac{2T}{G}L_3 + G\sin\theta\left[\cos\theta\left(\frac{\sin^2\varphi}{A_1} + \frac{\cos^2\varphi}{A_2} - \frac{1}{A_3}\right)\times \times (L_2\cos\psi - L_1\sin\psi) + \right. \quad (2.2)$$

$$\left. + \sin\varphi\cos\varphi\left(\frac{1}{A_1} - \frac{1}{A_2}\right)(L_1\cos\psi + L_2\sin\psi)\right]$$

The satellite center of mass moves in a circular orbit with revolution period Q . The true anomaly ν depends on time t as follows:

$$\nu = \omega_0 t,\ \omega_0 = \frac{2\pi}{Q} \quad (2.3)$$

Here, ω_0 is the angular velocity of orbital motion. We write the projections of the gravitational torque L_i^g and the external resistance torque L_i^r onto the axes Oy_i in the form introduced in [5, 7]. Here we present the projection onto the axis Oy_1 (the projections on the other axes can be written in a similar way):

$$L_1^g = 3\omega_0^2\sum_{j=1}^{3}(\beta_2\beta_j S_{3j} - \beta_3\beta_j S_{2j}) \quad (2.4)$$

$$L_1^r = -G\sum_{i=1}^{3}\left(\frac{I_{i1}\alpha_{1i}\alpha_{31}}{A_1} + \frac{I_{i2}\alpha_{1i}\alpha_{32}}{A_2} + \frac{I_{i3}\alpha_{1i}\alpha_{33}}{A_3}\right) \quad (2.5)$$

$$S_{mj} = \sum_{p=1}^{3}A_p\alpha_{jp}\alpha_{mp}, \beta_1 = \cos(\nu-\lambda)\cos\delta$$

$$\beta_2 = \sin(\nu-\lambda),\ \beta_3 = \cos(\nu-\lambda)\sin\delta$$

Projections of the torque of forces of highly viscous fluid in a cavity L_i^p onto the axes Oy_i $(i = 1,2,3)$ have the following form [3]:

$$L_i^p = \frac{P}{A_1A_2A_3}\left\{\omega\cdot\mathbf{B}^i + 3\omega_0^2(\mathbf{D}+\mathbf{S})\cdot\alpha^i\right\},\ i = 1,2,3 \quad (2.6)$$

$$\omega = \begin{pmatrix} p \\ q \\ r \end{pmatrix}, \ \mathbf{B}^i = \begin{pmatrix} B_1^i \\ B_2^i \\ B_3^i \end{pmatrix}, \ \alpha^i = \begin{pmatrix} \alpha_{i1} \\ \alpha_{i2} \\ \alpha_{i3} \end{pmatrix}, \ \alpha^* = \frac{1}{1-\alpha_{33}^2}$$

$$\mathbf{D} = \begin{pmatrix} A_2A_3\left(A_3-A_2\right)\left[-\gamma_{31}\gamma_{33}r+\alpha^*\left(F_1p_{\alpha1}+M_1p_{\alpha2}\right)\right] \\ A_1A_3\left(A_1-A_3\right)\left[-\gamma_{32}\gamma_{33}r+\alpha^*\left(F_2p_{\alpha1}+M_2p_{\alpha2}\right)\right] \\ \left(A_3-A_2\right)\left[\left(\gamma_{32}^2-\gamma_{31}^2\right)r-\alpha^*\left(F_3p_{\alpha1}+M_3p_{\alpha2}\right)\right] \end{pmatrix}$$

$$\mathbf{F} = \begin{pmatrix} \gamma_{31}\gamma_{33}\alpha_{33}+\beta_{\alpha1}\gamma_{33}+\beta_{\alpha2}\gamma_{32} \\ \gamma_{32}\gamma_{33}\alpha_{33}+\beta_{\alpha3}\gamma_{33}+\beta_{\alpha2}\gamma_{31} \\ \left(\gamma_{32}^2-\gamma_{31}^2\right)\alpha_{33}+\beta_{\alpha3}\gamma_{32}+\beta_{\alpha1}\gamma_{31} \end{pmatrix}$$

$$\mathbf{M} = \begin{pmatrix} \gamma_{32}^2\alpha_{32}+\gamma_{32}\left(\gamma_{33}\alpha_{33}-\upsilon_3\right) \\ \gamma_{33}^2\alpha_{31}+\gamma_{31}\left(\gamma_{33}\alpha_{33}-\upsilon_3\right) \\ \gamma_{33}\left(\gamma_{32}\alpha_{31}+\gamma_{31}\alpha_{32}\right) \end{pmatrix}$$

$$\mathbf{S} = \begin{pmatrix} \gamma_{31}\left[\gamma_{33}rA_3(A_1A_2-A_1^2-A_2A_3+A_3^2)+Q_1)\right] \\ \gamma_{32}\left[\gamma_{31}pA_1(A_3A_2-A_2^2-A_1A_3+A_1^2)+Q_2\right] \\ \gamma_{33}\left[\gamma_{32}qA_2(A_1A_3-A_3^2-A_1A_2+A_2^2)+Q_3\right] \end{pmatrix}$$

$$Q_1 = \gamma_{32}qA_2(A_1A_3-A_1^2-A_2A_3+A_2^2)$$

$$Q_2 = \gamma_{33}rA_3(A_1A_2-A_2^2-A_1A_3+A_3^2)$$

$$Q_3 = \gamma_{31}pA_1(A_2A_3-A_3^2-A_1A_2+A_1^2)$$

$$\gamma_{3i} = \beta_1\alpha_{1i}+\beta_2\alpha_{2i}+\beta_3\alpha_{3i}, \ i=1,2,3$$

$$p_{\alpha1} = p\alpha_{31}+q\alpha_{32}, \ p_{\alpha2} = p\alpha_{32}+q\alpha_{31}$$

$$\upsilon_{\alpha1} = -\alpha_{22}\upsilon_1+\alpha_{12}\upsilon_2, \ \upsilon_{\alpha2} = -\alpha_{23}\upsilon_1+\alpha_{13}\upsilon_2$$

$$\upsilon_{\alpha3} = -\alpha_{21}\upsilon_1+\alpha_{11}\upsilon_2$$

$$B_1^i = \left[\omega_2^2A_2(A_1-A_2)(A_2-A_3+A_1)+\right.$$

$$\left.\omega_3^2A_3(A_1-A_3)(A_3-A_2+A_1)\right]\alpha_{i1}$$

Here, α_{ij} are direction cosines between coordinate systems Oy_i $(i=1,2,3)$ and Oz_i $(i=1,2,3)$, p, r, q are projections of the absolute angular velocity vector ω of the satellite relative to the $0x_1x_2x_3$ coordinate system onto the axes Oz_i $(i=1,2,3)$. Quantity $\tilde{\mathbf{P}}$ is a tensor depending on the cavity shape only, it characterizes the dissipative torque of forces, caused by viscous fluid, in the quasi-static approximation [3]. For the sake of simplicity, in (2.6) the so called scalar tensor is considered, which is determined by a single scalar quantity $P>0$. The components of this tensor are $\tilde{\mathbf{P}}_{ij} = P\delta_{ij}$, where δ_{ij} are Kronecker‘s symbols (tensor $\tilde{\mathbf{P}}$ has such a form, for example, in the case of spherical cavity). The dynamically asymmetric satellite is considered, whose moments of inertia, for certainty, satisfy the inequality $A_1>A_2>A_3$. We assume that the angular velocity ω of the satellite motion about the center of mass is significantly larger than the angular velocity ω_0 of the orbital motion, i.e. $\varepsilon = \omega_0/\omega \sim A_1\omega_0/G << 1$.

In the present paper, we assume that the resistance torque $\mathbf{L}^r$ can be represented as $\mathbf{L}^r = \mathbf{I}\omega$ where the tensor $\mathbf{I}$ has constant components I_{ij} in the body-fixed frame [1, 7]. We assume that the medium resistance is weak and has the order of ε^2, $\|\mathbf{I}\|/G_0 \sim \varepsilon^2 << 1$, where $\|\mathbf{I}\|$ is the norm of matrix of the resistance coefficients and G_0 is the satellite angular momentum at the initial time. It is supposed in the paper, that the cavity is filled with a high-viscosity fluid, i.e. $\vartheta >> 1$ $\left(\vartheta^{-1} \sim \varepsilon\right)$, the cavity's shape is spherical, then [3]

$$\tilde{\mathbf{P}} = Pdiag(1,1,1),\;\; P = \frac{8\pi\rho a^7}{525\vartheta} \tag{2.7}$$

Here ρ and ϑ are the density and kinematical coefficient of viscosity of fluid in a cavity, respectively, a is the cavity radius.

Therefore, to an accuracy of quantities of the second order of smallness $\left(P \sim \varepsilon^2, \|\mathbf{I}\| / G_0 \sim \varepsilon^2\right)$, and the projections of the torque of forces of viscous fluid in a cavity have a form:

$$\begin{aligned} L_i^p = \frac{P}{A_1A_2A_3} \{ & p\left[q^2A_2(A_1-A_2)(A_2-A_3+A_1) + +r^2A_3(A_1-A_3)(A_3-A_2+A_1)\right]\alpha_{i1} + \\ & +q\left[r^2A_3(A_2-A_3)(A_3-A_1+A_2) + p^2A_1(A_1-A_2)(A_3-A_1-A_2)\right]\alpha_{i2} + \\ & +r\left[p^2A_1(A_3-A_1)(A_1-A_2+A_3) + \right. \\ & \left. +q^2A_2(A_3-A_2)(A_2-A_1+A_3)\right]\alpha_{i3}\},\; i=1,2,3 \end{aligned} \tag{2.8}$$

The problem is formulated to study the evolution of satellite rotations over an asymptotically large time interval $t \sim \varepsilon^{-2}$, on which the motion parameters essentially change.

3 Modified procedure of the averaging method

For the considered problem of solving system (2.1)-(2.3) at small ε over the time interval $t \sim \varepsilon^{-2}$ we apply the modified scheme of the averaging method [5, 19, 20]. Consider the unperturbed motion $(\varepsilon = 0)$, when the torques of applied forces are zero. In this case the rotation of the rigid body is an Euler-Poinsot motion. The variables G, δ, λ, T and ν become constants, and φ,ψ and θ are functions of time t. The slow variables in the perturbed motion are G, δ, λ, T and ν, while the fast variables are Euler angles φ, ψ, and θ. Consider the motion under the condition $2TA_1 \geq G^2 > 2TA_2$ corresponding the case in which the trajectories of the angular momentum vector surround the axis Oz_1 of the maximal moment of inertia A_1 [21]. We introduce the quantity

$$k^2 = \frac{(A_2-A_3)\left(2TA_1-G^2\right)}{(A_1-A_2)\left(G^2-2TA_3\right)} \quad \left(0 \leq k^2 \leq 1\right) \tag{3.1}$$

In the unperturbed motion, it is a constant, namely, the modulus of the elliptic functions describing this motion. For constructing the averaged system of the first approximation we substitute the solution of the unperturbed Euler-Poinsot motion [21] into the right-hand sides of equations of motion (2.1), (2.2) and perform averaging over the variable ψ and then over the time t with the dependencies of φ and θ on t [5] taken into account. The previous notation for the slow variables δ, λ, G and T is preserved. As a result, we get

$$\begin{aligned} \frac{dG}{dt} = -\frac{G}{R(k)} \{ & I_{22}(A_1-A_3)W(k) + I_{33}(A_1-A_2)\times \\ & \times\left[k^2 - W(k)\right] + I_{11}(A_2-A_3)\left[1-W(k)\right]\} \end{aligned} \tag{3.2}$$

$$\frac{dT}{dt} = -\frac{2T}{R(k)}\left\{I_{22}(A_1-A_3)W(k) + I_{33}(A_1-A_2)\left[k^2-W(k)\right] + \right.$$

$$+\frac{(A_1-A_2)(A_1-A_3)(A_2-A_3)}{S(k)}\left[\frac{I_{33}}{A_3}\left[k^2-W(k)\right]+\right.$$

$$\left.+\frac{I_{22}}{A_2}\left(1-k^2\right)W(k)\right]+\frac{I_{11}}{A_1}\frac{(A_2-A_3)R(k)}{S(k)}\left[1-W(k)\right]\Big\}-$$

$$-\frac{4PT^2(A_1-A_3)(A_1-A_2)(A_2-A_3)}{3A_1^2A_2^2A_3^2S^2(k)}\times$$

$$\times\left\{A_2(A_1-A_3)(A_1+A_3-A_2)\left[(k^2-1)+(1+k^2)(1-W(k))\right]+\right.$$

$$+A_1(A_2-A_3)(A_3+A_2-A_1)\left[(k^2-2)W(k)+k^2\right]+$$

$$\left.+A_3(A_1-A_2)(A_1+A_2-A_3)\left[(1-2k^2)W(k)+k^2\right]\right\}$$

$$\frac{d\delta}{dt}=-\frac{3\omega_0^2}{2G}\beta_2\beta_3N^*,\ \frac{d\lambda}{dt}=\frac{3\omega_0^2}{2G\sin\delta}\beta_1\beta_3N^*$$

$$N^*=A_2+A_3-2A_1+3\left(\frac{2A_1T}{G^2}-1\right)\left[A_3+(A_2-A_3)\frac{K(k)-E(k)}{K(k)k^2}\right]$$

$$W(k)=1-\frac{E(k)}{K(k)},S(k)=A_2-A_3+(A_1-A_2)k^2$$

$$R(k)=A_1(A_2-A_3)+A_3(A_1-A_2)k^2$$

Here, $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind, respectively [22]. By differentiating the expression for k^2 (3.1) and by using the first two equations in (3.2), we obtain the differential equation

$$\frac{dk^2}{dt}=\frac{S(k)}{T(A_1-A_2)(A_1-A_3)(A_2-A_3)}\left[R(k)\frac{dT}{dt}-S(k)G\frac{dG}{dt}\right] \tag{3.3}$$

The expression in braces on the right-hand side of the equation for G in (3.2) is positive (for $A_1>A_2>A_3$), because the inequalities $(1-k^2)K\le E\le K$ are satisfied [22]. The coefficient of each I_{ii} is a negative function of k^2, and moreover, all of them cannot be zero simultaneously. Since $G>0$, we have $\frac{dG}{dt}<0$; i.e. the variable G strictly decrease for any $k^2\in[0;1]$. The expressions in curly brackets of the right-hand side of (3.2) for T is positive (for $A_1>A_2>A_3$), because the inequalities $(1-k^2)K\le E\le K$ are valid. Therefore, $\frac{dT}{dt}<0$ since $T>0$, i.e., the variable T strictly decreases for any $k^2\in[0;1]$.

Consider the system consisting of the last two equations in system (3.2) and equation (2.3).

They can be written as

$$\dot\delta=\omega_0^2\Delta(\nu,\delta,\lambda),\ \dot\lambda=\omega_0^2\Lambda(\nu,\delta,\lambda),\nu=\omega_0t$$

Here Δ and Λ are the coefficients on the right-hand sides in the last two equations in (3.2), δ and λ are slow variables, and ν is a semi-slow variable.

We obtain a system of special form, which we solve by a modified averaging method [20]: After the averaging, we have

$$\dot\delta=0,\ \dot\lambda=\frac{3\omega_0^2N^*\cos\delta}{4G} \tag{3.4}$$

We note that the action of the applied forces does not change the angular velocity δ and that deviation of the vector **G** from vertical remains constant in this approximation. The numerical integration was performed for the initial conditions $G(0) = 1$, $k^2(0) = 0.99$, $\delta(0) = 0.785$ and $\lambda(0) = 0.785$ and for the following values of the principal central moments of inertia of the body: $A_1 = 8$, $A_2 = 5, 6, 7$, and $A_3 = 4$. The values of the principal central moments of inertia of the body are: $I_{11} = 2.322$, $I_{22} = 1.31$, $I_{33} = 1.425$ or $I_{11} = 2.6$, $I_{22} = 3.0$, $I_{33} = 0.5$.

The initial value of kinetic energy was found from the equality

$$T = \frac{G^2(0)\, S\left(k^2(0)\right)}{2R\left(k^2(0)\right)} \tag{3.5}$$

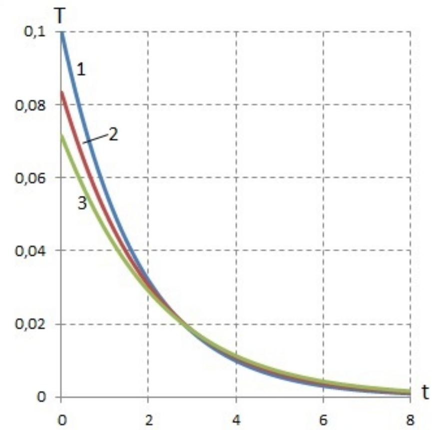

Fig. 2

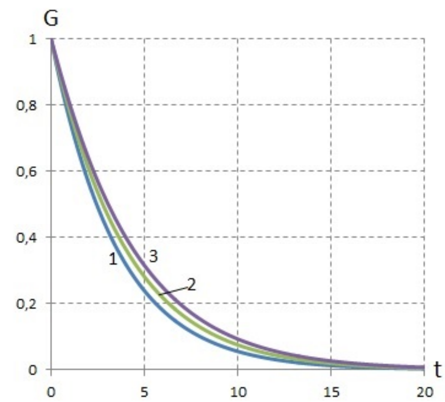

Fig. 3

The plots of kinetic energy change have the form presented in Fig. 2. Curves 1,2,3 correspond to various values of $A_2 = 5, 6, 7$. Numerical analysis shows that the functions $G(t)$ and $T(t)$ are monotone decreasing (Fig. 2, 3).This conclusions coincide with results of analytical discussions after the formula (3.3).

Figure 4 presents the plots of change of the angle λ of orientation of the angular momentum vector for various values of satellite's moments of inertia. Curves 1,2,3 correspond to various values of $A_2 = 7, 6, 5$ constant values $A_1 = 8$, $A_3 = 4$.The curvature of function $\lambda(t)$ grows when the value of the moment of inertia A_2 decreases.

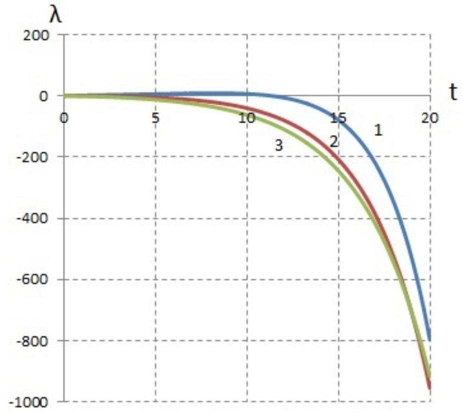

Fig. 4

4 Conclusion

The system obtained after averaging over the Euler-Poinsot motion and applying the modified averaging method, is analyzed. The analytical study and numerical analysis are performed. The orientation of the angular momentum vector in the orbital frame of reference is determined. In the approximation under study, the perturbed motion of the body consists of a fast Euler-Poinsot motion about the angular momentum vector and a slow evolution of the parameters of this motion. The angular momentum and the kinetic energy decrease and their variation depends only on the resistance torque. In the first approximation of the averaging method, the motion of the angular momentum vector **G** about the vertical on the orbital frame of reference is described by the last two equations of system (3.2). In the second approximation of the averaging method, the deviation of the angular momentum vector from vertical remains constant, and the angular velocity of rotation in this case is variable.

Acknowledgements

This study was supported by the third joint competition of Ukrainian State Foundation for Basic Research and Russian Foundation of Basic Research, 2013 year, project no. 953.1/ 010.

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