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EVOLUTION OF ROTATION OF A NEARLY DYNAMICALLY SPHERICAL TRIAXIAL SATELLITE UNDER THE ACTION OF LIGHT PRESSURE TORQUES

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The motion of a satellite relative to its center of mass under the action of torques of various (gravitational, magnetic, light pressure, etc.) nature has been investigated in numerous papers (see [1–6] and the bibliography therein). The estimate [1] of the perturbing torques shows that at heights exceeding 35,000–40,000 km above the Earth surface the light pressure torques substantially influence spacecraft motion. Using the averaging method, we investigate the attitude motion due to light pressure torques of a nearly dynamically spherical spacecraft whose shape is a surface of revolution. This case is of interest both theoretically and in applications.

1. BASIC ASSUMPTIONS AND STATEMENT OF THE PROBLEM

Consider the motion of a spacecraft relative to its center of mass under the action of the light pressure torques. We introduce three right-handed Cartesian coordinate frames with origins at the satellite center of mass [1, 2]. The coordinate frame $OXYZ$ moves translationally along the Sun orbit together with the satellite; the Y-axis is normal to the orbit plane, the Z-axis is parallel to the position vector of the orbit perihelion, and the X-axis is parallel to the velocity of the satellite center of mass at the perihelion. We describe the orientation of the angular momentum vector $\mathbf{L}$ in the reference frame $OXYZ$ by the angles ρ and σ as shown in [1, 2, 4, 6]. To construct the reference frame OL_1L_2L associated with the vector $\mathbf{L}$, in the plane OYL we draw an axis L_1 perpendicular to the vector $\mathbf{L}$ and forming an obtuse angle with the Y-axis. The L_2-axis completes the L- and L_1-axes to a right-handed coordinate frame. The axes of the satellite-connected coordinate frame $Oxyz$ coincide with the principal central axes of inertia of the satellite. We describe the relative position of the principal central axes of inertia and the axes L, L_1, and L_2 by Euler's angles φ, ψ, and θ [1, 2, 4, 6]. The direction cosines α_{ij} of the Ox-, Oy-, and Oz-axes in the coordinate frame OL_1L_2L can be expressed via the Euler angles φ, ψ, and θ according to the well-known formulas [1].

We assume that the spacecraft moves around the Sun along an elliptic orbit and the moments of all forces, apart from the light pressure forces, are negligible. Moreover, we assume that the satellite surface is a surface of revolution, with the unit vector $\mathbf{k}$ of the symmetry axis being directed along the Oz-axis. As is shown in [1, 3, 5], in this case the light pressure torque $\mathbf{M}$ acting on the satellite is given by

$$\mathbf{M} = a_c(\varepsilon_s)\frac{R_0^2}{R^2}\mathbf{e}_r \times \mathbf{k}, \tag{1.1}$$

where

$$a_c(\varepsilon_s)\frac{R_0^2}{R^2} = p_c S(\varepsilon_s) z_0'(\varepsilon_s), \quad p_c = \frac{E_0}{c}\frac{R_0^2}{R^2}.$$

Here $\mathbf{e}_r$ is the unit vector codirected with the position vector of the satellite center of mass; ε_s is the angle between $\mathbf{e}_r$ and $\mathbf{k}$, so that $|\mathbf{e}_r \times \mathbf{k}| = \sin\varepsilon_s$; R is the current distance from the Sun center to the satellite center of mass; R_0 is some fixed value of R, for example, at the initial instant; $a_c(\varepsilon_s)$ is the light pressure torque coefficient; S is the shadow area on the plane normal to the flux; z_0' is the distance between the center of mass and the pressure center; p_c is the light pressure at the distance R from the Sun center; c is the light velocity; E_0 is the light energy flux at the distance R_0 from the Sun center. If R_0 is the Earth orbit radius, then $p_{c0} = 4.64 \cdot 10^{-6}\,\mathrm{N/m^2}$.

We assume in addition that $a_c = a_c(\cos\varepsilon_s)$ [1] and approximate the function a_c by polynomials in $\cos\varepsilon_s$. The light pressure torques have a force function that depends only on the orientation of the symmetry axis of the body [1]. Let us represent the force function $a_c(\cos\varepsilon_s)$ in the form

$$a_c = a_{0c} + 2a_{1c}\cos\varepsilon_s. \tag{1.2}$$

If the force function exists, then the equations of the satellite perturbed motion in the variables L, ρ, σ, φ, ψ, and θ have the form [2, 4]

$$\begin{aligned}
&\dot\sigma = \frac{1}{L\sin\rho}\frac{\partial U}{\partial \rho}, \quad \dot\rho = -\frac{1}{L\sin\rho}\frac{\partial U}{\partial\sigma} + \frac{\cot\rho}{L}\frac{\partial U}{\partial\psi}, \quad \dot L = \frac{\partial U}{\partial \psi},\\
&\dot\theta = L\sin\theta\sin\varphi\cos\varphi\left(\frac{1}{A} - \frac{1}{B}\right) - \frac{1}{L\sin\theta}\frac{\partial U}{\partial\varphi} + \frac{\cot\theta}{L}\frac{\partial U}{\partial\psi},\\
&\dot\varphi = L\cos\theta\left(\frac{1}{C} - \frac{\sin^2\varphi}{A} - \frac{\cos^2\varphi}{B}\right) + \frac{1}{L\sin\theta}\frac{\partial U}{\partial\theta},\\
&\dot\psi = L\left(\frac{\sin^2\varphi}{A} + \frac{\cos^2\varphi}{B}\right) - \frac{1}{L}\left(\frac{\partial U}{\partial\rho}\cot\rho + \frac{\partial U}{\partial\theta}\cot\theta\right).
\end{aligned} \tag{1.3}$$

The force function U depends on time t via the true anomaly $\nu(t)$ and on the direction cosines α_3, β_3, and γ_3 of the Oz-axis in the coordinate frame $OXYZ$; thus, $U = U(\nu(t), \alpha_3, \beta_3, \gamma_3)$.

System (1.3) must be completed by the equation

$$\frac{\partial\nu}{\partial t} = \frac{(1+e\cos\nu)^2}{(1-e^2)^{3/2}}\omega_0, \qquad \text{where} \quad \omega_0 = \frac{2\pi}{T_0} = \sqrt{\frac{\kappa(1-e^2)^3}{P^3}}, \tag{1.4}$$

describing the evolution of the true anomaly in time. Here ω_0 is the mean angular velocity of the center-of-mass motion along the elliptic orbit; T_0 is the period of the satellite revolution; e and P are, respectively, the eccentricity and the focal parameter of the orbit; κ is the product of the universal gravitational constant by the Sun mass.

The torque in Eq. (1.1) corresponds to the force function

$$U(\cos\varepsilon_s) = -\frac{R_0^2}{R^2}\int a_c(\cos\varepsilon_s)\, d(\cos\varepsilon_s).$$

Consider the expression (1.2) for $a_c(\cos\varepsilon_s)$. In this case the force function has the form

$$U(\cos\varepsilon_s) = -\frac{R_0^2}{R^2}(a_{0c}\cos\varepsilon_s + a_{1c}\cos^2\varepsilon_s), \tag{1.5}$$

where $\cos\varepsilon_s = \gamma_3\cos\nu + \alpha_3\sin\nu$ and α_3 and γ_3 are expressed via ρ, σ, θ, and ψ by the well-known formulas [1].

Suppose that the satellite principal central moments of inertia are nearly the same and can be represented in the form

$$A = J_0 + \varepsilon A', \quad B = J_0 + \varepsilon B', \quad C = J_0 + \varepsilon C', \tag{1.6}$$

where ε is a small parameter ($0 < \varepsilon \ll 1$). Assume in addition that $a_{0c} \sim \varepsilon$ and $a_{1c} \sim \varepsilon$, that is, the light pressure torques and the gyroscopic torques are both of the order of ε. It follows from (1.5) that $U \sim \varepsilon$. We investigate the solution of system (1.3), (1.4) for small ε on a large time interval $t \sim \varepsilon^{-1}$. The error of the averaged solution for the slow variables is $O(\varepsilon)$ on the time interval on which the body performs $\sim \varepsilon^{-1}$ revolutions. The averaging over ψ and ν is performed independently, just as for the nonresonance cases [2].

2. THE AVERAGING SCHEME AND THE CONSTRUCTION OF THE FIRST APPROXIMATION SYSTEM

Consider the unperturbed motion ($\varepsilon = 0$). In this case equations (1.3) and (1.4) describe the motion of a spherically symmetric body, and the light pressure torque (1.1) is zero. In this case system (1.3) implies that σ, ρ, L, θ, and φ are constant, whereas

$$\psi = \frac{L}{J_0}t + \psi_0, \qquad \psi_0 = \text{const}, \tag{2.1}$$

which corresponds to the uniform rotation of the satellite about the angular momentum vector **L**, which moves translationally. For small $\varepsilon \neq 0$, the variables σ, ρ, L, θ, and φ are slow variables, and ψ and ν are fast variables in the system of seven equations (1.3) and (1.4), provided that relations (1.6) hold. To obtain the solution in the first approximation, it suffices to average the right-hand sides of Eqs. (1.3) in which ν is the solution of Eq. (1.4) and ψ is defined in (2.1). We assume that the frequencies ω_0 and L/J_0 satisfy the condition $m\omega_0 + nL/J_0 \neq 0$ for any integer m and n. As is shown in [2], under this assumption the averaging with respect to time can be replaced by independent averaging with respect to ψ and ν, since these variables are functions of t. With allowance for (1.4), the time averaging of functions of ν is reduced to the averaging with respect to ν as follows:

$$M_t\{f(\nu)\} = \frac{1}{T}\int_0^T f(\nu)\,dt = \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-e^2)^{3/2} f(\nu)}{(1+e\cos\nu)^2}\,d\nu = (1-e^2)^{3/2} M_\nu\left\{\frac{f(\nu)}{(1+e\cos\nu)^2}\right\}. \tag{2.2}$$

Using the well-known expressions for the direction cosines α_3, β_3, and γ_3 of the Oz-axis with respect to the coordinate frame $OXYZ$ [1], we obtain the average of the force function over ψ:

$$\langle U\rangle_\psi = -a_{1c}\frac{R_0^2}{R^2}\left[\left(1-\tfrac{3}{2}\sin^2\theta\right)\sin^2\rho\cos^2(\sigma-\nu) + \tfrac{1}{2}\sin^2\theta\right]. \tag{2.3}$$

On averaging over ν according to (2.2) with allowance for the equation of motion of the satellite center of mass along an elliptic orbit $R = P(1+e\cos\nu)^{-1}$, the force function acquires the form

$$U_0 = -\frac{1}{2}(1-e^2)^{3/2} a_{1c}\frac{R_0^2}{P^2}\left[\left(1-\tfrac{3}{2}\sin^2\theta\right)\sin^2\rho + \sin^2\theta\right]. \tag{2.4}$$

Calculating the partial derivatives $\partial U_0/\partial\rho$ and $\partial U_0/\partial\theta$ and taking into account the identities $\partial U_0/\partial\sigma = \partial U_0/\partial\psi = \partial U_0/\partial\varphi \equiv 0$, we arrive at the first approximation system for the slow variables in the form

$$\begin{gathered}\dot\sigma = -(1-e^2)^{3/2} a_{1c}\frac{R_0^2}{LP^2}\left(1-\tfrac{3}{2}\sin^2\theta\right)\cos\rho, \quad \dot\rho = 0, \quad \dot L = 0, \quad \dot\theta = L\sin\theta\sin\varphi\cos\varphi\left(\frac{1}{A}-\frac{1}{B}\right), \\ \dot\varphi = \cos\theta\left[L\left(\frac{1}{C}-\frac{\sin^2\varphi}{A}-\frac{\cos^2\varphi}{B}\right) - (1-e^2)^{3/2} a_{1c}\frac{R_0^2}{LP^2}\left(1-\tfrac{3}{2}\sin^2\rho\right)\right].\end{gathered} \tag{2.5}$$

Note that the coefficient a_{0c} disappears on averaging. Now we must study system (2.5). The vector of angular momentum remains constant in magnitude and inclined at a constant angle with respect to the normal to the orbit plane.

Consider Eqs. (2.5) for θ and φ. They describe the motion of the vector **L** of the angular momentum relative to the body.

3. EVOLUTION OF THE RIGID BODY ROTATIONS

3.1. Analytical investigation. Equations (2.5) for θ and φ in the slow time ξ can be reduced to the form

$$\theta' = \sin\theta\sin\varphi\cos\varphi, \qquad \varphi' = \cos\theta\,(\mu - \sin^2\varphi), \tag{3.1}$$

where

$$\xi = L_0\beta t, \quad \mu = \frac{\alpha-\gamma}{\beta}, \quad \beta = \frac{1}{A}-\frac{1}{B}, \quad \gamma = \frac{1}{B}-\frac{1}{C}, \quad \alpha = -(1-e^2)^{3/2} a_{1c}\frac{R_0^2}{L_0^2 P}\left(1-\tfrac{3}{2}\sin^2\rho_0\right).$$

Here L_0 is the value of L at the initial time instant. Taking into account (1.6) and the assumption that $a_{1c} \sim \varepsilon$, we find that β, γ, and α are $O(\varepsilon)$. System (3.1) has the first integral

$$\sin^2\theta\,(\mu - \sin^2\varphi) = c_1 = \sin^2\theta_0\,(\mu - \sin^2\varphi_0) = \text{const}. \tag{3.2}$$

If the light pressure torque is lacking, then $a_{0c} = 0$, $a_{1c} = 0$, and system (3.1) is reduced to the form

$$\theta' = \sin\theta\sin\varphi\cos\varphi, \qquad \varphi' = \cos\theta\,(\mu^* - \sin^2\varphi), \qquad \text{where} \quad \mu^* = -\gamma/\beta. \tag{3.3}$$

To be definite, let us assume that $A > B > C$; then $\beta < 0$, $\gamma < 0$, and $\mu < 0$. Let us introduce the variable $x = \cos\theta$. Then equation (3.1) can be transformed, with allowance for (3.2), to an equation that admits separation of variables. The subsequent integration yields

$$\eta(t - t_0) = \int_{x_0}^{x} \frac{dx_1}{\sqrt{(x_1^2 - h)(a^2 - x_1^2)}}, \tag{3.4}$$

where

$$\eta = L_0\left(\frac{1}{B} - \frac{1}{A}\right)\sqrt{\mu(\mu - 1)}, \quad h = 1 - \frac{c_1}{\mu}, \quad a^2 = 1 - \frac{c_1}{\mu - 1}.$$

Thus, the problem is reduced to a quadrature. The integral on the right-hand side in (3.4) is an elliptic integral. To obtain the solution to (3.1), one must first invert the integral in (3.4). This inversion can be performed in various ways, depending on the values of the roots of the radicand in the integral (3.4).

If we assume $C > B > A$, then $\beta > 0$ and $\gamma > 0$; $\mu > 0$ if $\alpha > \gamma$ and $\mu < 0$ if $\alpha < \gamma$. These cases are considered quite similarly. First, consider first some inequalities which are satisfied by the parameters h and a^2 in (3.4). One can show that $h = 1 - c_1/\mu \le 1$, $h \le a^2$, and $a^2 \le 1$. These inequalities can be satisfied both for $h \le 0$ and for $h \ge 0$.

Let us invert the integral in (3.4) for the case $h \le 0$. Since $|x| \le a$ in this case, we make the change of variable $x = a\cos\chi$. Then relation (3.4) can be reduced to the form

$$\tau = \lambda(t - t_*) = \int_0^{\chi} \frac{d\chi}{\sqrt{1 - k^2\sin^2\chi}}, \tag{3.5}$$

where

$$\lambda = \eta\sqrt{a^2 - h}, \quad k^2 = \frac{a^2}{a^2 - h} \le 1.$$

Here t_* is some fixed time instant. Thus, we have arrived at the elliptic integral of the first kind. The inversion of this integral yields [7]

$$\chi = \operatorname{am}\tau, \quad \cos\chi = \operatorname{cn}\tau, \quad \cos\theta = a\operatorname{cn}\tau. \tag{3.6}$$

It follows from the last relation in (3.6) that $\operatorname{am}(\cos\theta) = a$. The functions $\operatorname{cn}\tau$ and $\operatorname{sn}\tau$ are periodic with period $T_\tau = 4\mathbf{K}(k^2)$, where $\mathbf{K}(k^2)$ is the complete elliptic integral of the first kind. Obviously, the oscillation period of the angle θ is $T_\theta = 4\mathbf{K}(k^2)/\lambda$. Thus, the angle θ as a function of time is expressed in terms of Jacobi's elliptic functions.

To determine the function $\varphi(t)$, it suffices to know time histories of the functions $\sin\varphi\sin\theta$ and $\cos\varphi\sin\theta$. Using the first integral (3.2), we obtain (for definiteness, we take the sign plus when extracting the square root [4])

$$\sin\varphi\sin\theta = \sqrt{\mu(h - a^2)}\,\operatorname{dn}\tau, \quad \cos\varphi\sin\theta = \sqrt{1 - \mu}\,a\operatorname{sn}\tau. \tag{3.7}$$

Thus, all direction cosines of the angular momentum vector with respect to the principal central axes of inertia are T_θ-periodic functions.

Let us now integrate Eq. (3.1) for $h \ge 0$. We set $h = b^2$ and represent (3.4) as follows

$$\eta(t - t_0) = \int_x^a \frac{dx_1}{\sqrt{(a^2 - x_1^2)(x_1^2 - b^2)}}. \tag{3.8}$$

Let us make the change of variables $x_1^2 = a^2\cos^2\chi + b^2\sin^2\chi$. After some manipulations, the integral (3.8) is reduced to the form (3.5), where $\lambda \to a\eta$, $t_* \to t_0$, and the modulus k is specified by the relation $0 \le k^2 = (a^2 - b^2)/a^2 < 1$. The inverse of the latter integral is $\chi = \operatorname{am}\tau$, $\tau = a\eta(t - t_0)$. Then

$$x = \cos\theta = a\operatorname{dn}\tau. \tag{3.9}$$

Using the first integral (3.2), we obtain (up to the sign)

$$\sin\varphi\sin\theta = \sqrt{\mu(b^2 - a^2)}\,\operatorname{cn}\tau, \quad \cos\varphi\sin\theta = \sqrt{(1 - \mu)(a^2 - b^2)}\,\operatorname{sn}\tau. \tag{3.10}$$

Thus, in the first approximation of the averaging method, one can see an analogy between the problem in question and the Euler–Poinsot case; the only difference is in the coefficients μ and μ^*. In the slow time τ, the problem on the motion of a nearly dynamically spherical rigid body under the action of the light pressure torque is equivalent to

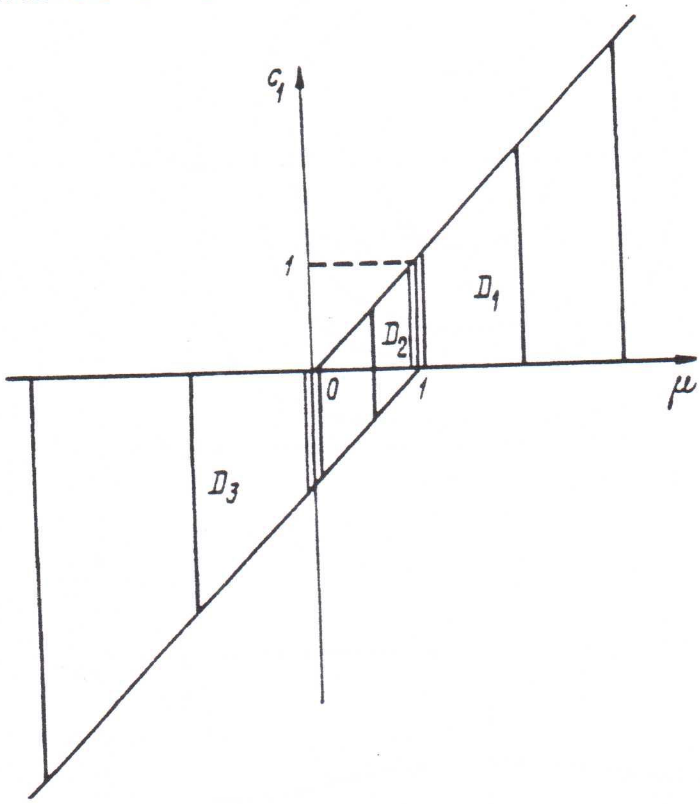

Fig. 1

the problem on the motion of an imaginary rigid body with arbitrary moments of inertia. This is accounted for by the adopted approximation $a_c = a_{0c} + 2a_{1c}\cos\varepsilon_s$ and is the main qualitative result of the investigation.

3.2. Qualitative analysis of the phase plane (θ, φ). Let us investigate system (3.1) for θ and φ with the first integral c_1 (3.2). The ranges of the variables θ and φ in this system are $0 \le \theta \le \pi$ and $0 \le \varphi < 2\pi$; the parameter μ can assume arbitrary values, $-\infty < \mu < +\infty$, depending on the relationships between the moments of inertia. The admissible domain D for the parameters c_1 and μ is presented in Fig. 1. We divide the domain D into three subdomains, D_1, D_2, and D_3. The subdomain D_1 is specified by the inequalities $\mu \ge c_1 \ge 0$ $(\mu \ge 1)$; the subdomain D_2, by the inequalities $\mu \ge c_1 \ge \mu - 1$ $(0 \le \mu \le 1)$; the subdomain D_3, by the inequalities $0 \ge c_1 \ge \mu - 1$ $(\mu \le 0)$. The domain $D = D_1 \cup D_2 \cup D_3$, shown in Fig. 1, is the set of points between two broken lines: the upper boundary consists of the negative abscissa semi-axis and the bisector of the first quadrant; the lower boundary consists of the third quadrant bisector shifted right by unity and the ray $\mu \ge 1$.

The boundaries of the subdomains D_1, D_2, and D_3 are singular sets for system (3.1). The motion corresponding to domains D_1 and D_3 is oscillatory in θ and oscillatory or rotational in φ. The separatrix for the domain D_1 is given by $\sin^2\theta = (\mu - 1)(\mu - \sin^2\varphi)^{-1}$, and for the domain D_3, by $\sin^2\theta = \mu(\mu - \sin^2\varphi)^{-1}$. In the domain D_2, oscillations occur both in θ and in φ.

There are 11 distinctive cases for the choice of the parameter μ: 1) $\mu = 0$, 2) $\mu = 1$, 3) $\mu = +\delta$, 4) $\mu = -\delta$, 5) $\mu = 1 + \delta$, 6) $\mu = 1 - \delta$, 7) $\mu \ll -1$, 8) $\mu \gg +1$, 9) $\mu \le -1$, 10) $\mu \ge +1$, 11) $\mu \simeq \frac{1}{2}$ $(0 < \delta \ll 1)$.

Figure 2 displays the family of graphs of θ versus φ, which were obtained numerically on the basis of (3.2) for $\mu = 0$ (case 1). These graphs correspond to oscillations for various initial conditions. For $\mu = 1$ (case 2), the graphs of θ versus φ are obtained by shifting Fig. 2 by $\pi/2$ along the φ-axis. Figure 3 shows the graphs of θ versus φ obtained numerically on the basis of the first integral (3.2), for $\mu = -1.7$ (case 9). According to these graphs, only oscillations occur in the variable θ; in the variable φ, oscillations occur within the separatrix $\sin^2\theta = \mu(\mu - \sin^2\varphi)^{-1}$ and rotations outside this separatrix. For $\mu > 1$ (cases 5, 8, and 10), the graphs have the same shape but are shifted by $\pi/2$ along the φ-axis. As $\mu \to -\infty$ (case 7), the curves $\theta(\varphi)$ degenerate into parallel lines. Figure 4 shows the graphs of θ versus φ for $\mu = 0.95$ (case 6); in this case only oscillations in θ and φ are possible. The graphs of $\theta(\varphi)$ for $\mu = +\delta$ (case 3) are obtained by an appropriate deformation of Fig. 4. Figure 5 shows the graphs corresponding to the oscillatory motions for $\mu = 0.5$ (case 11).

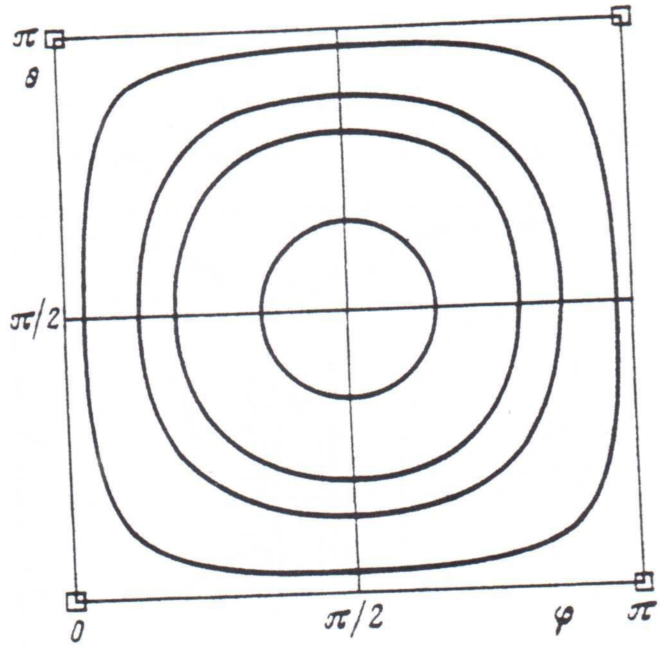

Fig. 2

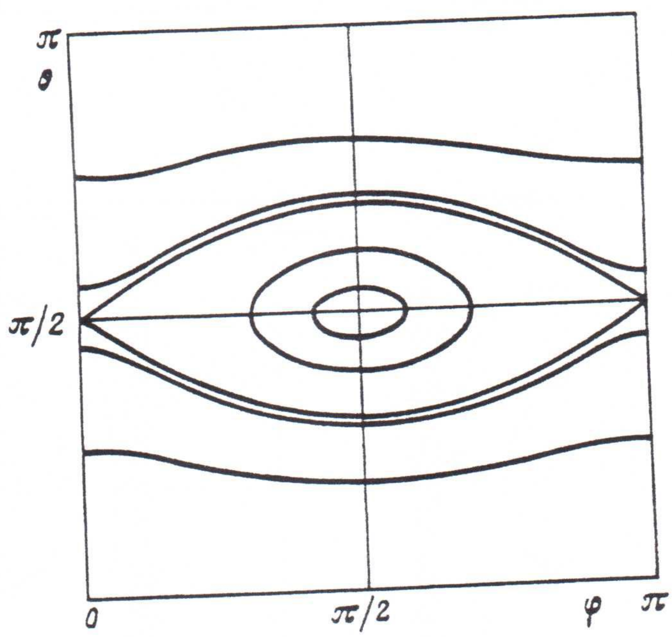

Fig. 3

4. INVESTIGATION OF THE EVOLUTION OF THE ANGULAR MOMENTUM IN THE ORBITAL FRAME

For $h \leq 0$, Eq. (2.5) for σ with allowance for (3.6) can be reduced to the form

$$\frac{d\sigma}{d\tau} = d(1 - 3a^2 \,\mathrm{cn}^2\, \tau), \tag{4.1}$$

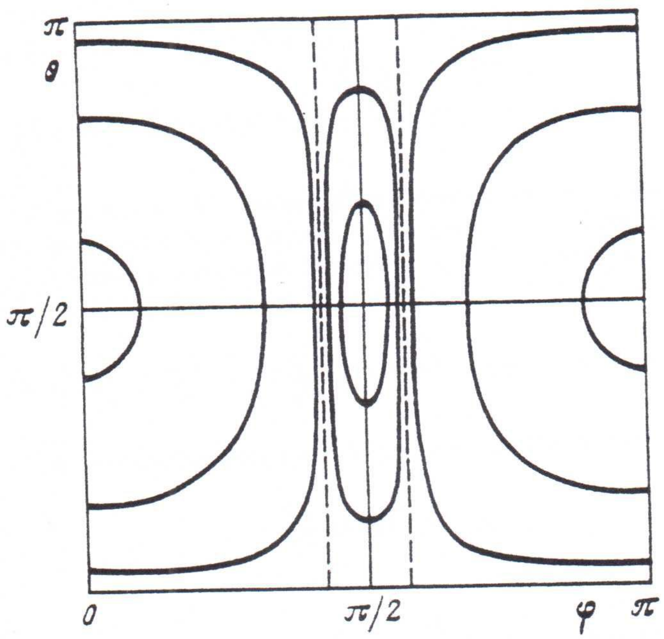

Fig. 4

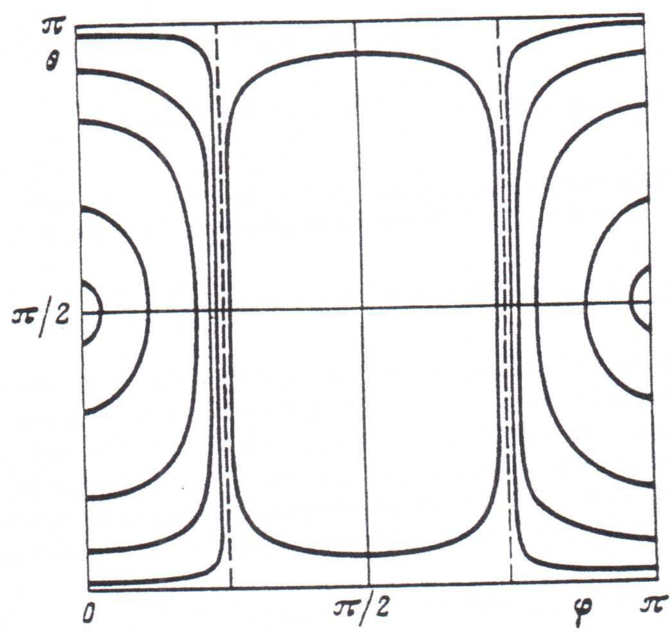

Fig. 5

where

$$\tau = \lambda(t - t_*), \quad d = \frac{1}{2}(1 - e^2)^{3/2} a_{1c} \frac{R_0^2}{L_0^2} \frac{AB}{A - B} \frac{1}{\sqrt{-c_1}} \cos \rho_0.$$

Here ρ_0 is the initial value of ρ. On integrating Eq. (4.1), we obtain

$$\sigma = \sigma_0 + d\left[\tau\left(1 + 3\frac{a^2 k'^2}{k^2}\right) - 3E(g, k)\frac{a^2}{k^2}\right], \tag{4.2}$$

where $E(g,k)$ is the elliptic integral of the second kind, k is the modulus of the elliptic function, k' is an additional modulus, σ_0 is the initial value of σ, and $g = \mathrm{am}\,\tau$ is the elliptic amplitude.

For small k, one can use series expansions of $E(g,k)$ [7]. By substituting these into (4.2), we obtain

$$\sigma = \sigma_0 + d\left\{ \frac{2\mathbf{K}y}{\pi}\left[1 + 3\frac{a^2}{k^2}\left(k'^2 - \frac{\mathbf{E}}{\mathbf{K}}\right)\right] - 3a^2\left[\left(1 - \tfrac{1}{4}k^2\right)(2\sin 2y + k^2 \sin 4y) \right.\right.$$
$$\left.\left. + \tfrac{1}{4}\left[\left(1+k^2\right)\sin y + k^2 \sin 3y\right]\left[\left(1-k^2\right)\cos y + k^2\cos 3y\right]\left(1 - \tfrac{1}{2}k^2\right)\right]\right\} + O\left(k^4\right), \tag{4.3}$$

where $y = \pi\tau(2\mathbf{K})^{-1}$; $\mathbf{K}$ and $\mathbf{E}$ are the complete elliptic integrals of the first and the second kind, respectively. Formula (4.3) is valid for any y and small k and consists of a linear and a oscillatory term in t.

For k close to unity and small g, one can use expansions of $E(g,k)$ given in [8]. Retaining the terms up to $O(k'^2)$ in these expansions and substituting them into (4.2), we obtain

$$\sigma = \sigma_0 + d\left\{\tau\left[1 + 3\frac{a^2k'^2}{k^2}\right] - 3\frac{a^2}{k^2}\left[\left(1 - \tfrac{1}{4}k'^2\right)\ln\tan\left(\tfrac{1}{2}\,\mathrm{am}\,\tau + \tfrac{\pi}{4}\right) + \tfrac{1}{4}k'^2\frac{\mathrm{sn}\,\tau}{\mathrm{cn}^2\,\tau}\right]\right\}. \tag{4.4}$$

For $h \geq 0$, Eq. (4.1) for σ with allowance for (3.9) can be reduced to the form

$$\frac{d\sigma}{d\tau} = N(1 - 3a^2\,\mathrm{dn}^2\,\tau), \tag{4.5}$$

where

$$\tau = a\eta(t - t_0), \quad N = (1-e^2)^{3/2}a_{1c}\frac{R_0^2}{L_0}\frac{AB}{A-B}\frac{1}{\sqrt{\mu(\mu - 1 - c_1)}}\cos\rho_0.$$

Integrating Eq. (4.5) yields

$$\sigma = \sigma_0 + N[\tau - 3a^2E(g,k)]. \tag{4.6}$$

For small k, using the expansion of $E(g,k)$ from [7], we find

$$\sigma = \sigma_0 + N\left[\frac{2\mathbf{K}y}{\pi}\left(1 - 3a^2\frac{\mathbf{E}}{\mathbf{K}}\right) - \frac{51}{8}a^2k^2\sin 2y\right] + O\left(k^4\right), \tag{4.7}$$

where $y = \pi\tau/(2\mathbf{K})$. Formula (4.7) is valid for any y and small k and contains a linear and an oscillatory term.

For k close to unity and small g, by using the expansion of $E(g,k)$ from [8], we obtain

$$\sigma = \sigma_0 + N\left\{\tau - 3a^2\left[\left(1 - \tfrac{1}{4}k'^2\right)\ln\tan\left(\tfrac{1}{2}\,\mathrm{am}\,\tau + \tfrac{\pi}{4}\right) + \tfrac{1}{4}k'^2\frac{\mathrm{sn}\,\tau}{\mathrm{cn}^2\,\tau}\right]\right\}. \tag{4.8}$$

As follows from Eq. (2.5) for σ, for θ close to 0 or π the velocity $\dot\sigma$ is negative if $\cos\rho > 0$ and positive if $\cos\rho < 0$. In the general case the motion with respect to the variable σ can be either oscillatory or rotational. If θ varies strongly, then the sign of the expression $1 - \frac{3}{2}\sin^2\theta$ can change. As a result, the value of σ can be practically constant for $\langle 1 - \frac{3}{2}\sin^2\theta\rangle = 0$. An analysis shows that there exist such values of the parameters μ, θ_0, and φ_0 for which $\sigma = \mathrm{const}$.

In the problem in question the character of the motion evolution is more complicated compared with the case of dynamically symmetric satellite ($A = B \neq C$), since in the former case the number of slow variables is greater by unity.

5. SPECIAL CASES OF THE BODY MOTION

The value $\theta = 0$ is a stationary point of Eq. (3.1) for θ. For $\theta = 0$, the differential equation for φ admits separation of variables. On integrating it for $\mu > 1$, we obtain the expression

$$\tan\varphi = l\tan[r\xi + \arctan(l^{-1}\tan\varphi_0)], \tag{5.1}$$

where

$$l = \sqrt{\frac{\mu}{\mu - 1}}, \quad r = \sqrt{\mu(\mu-1)}, \quad \xi = L_0\beta t.$$

For $\mu < 0$, the solution to Eq. (3.1) for φ ($\theta = 0$) acquires the form

$$\tan\varphi = l\tan[-r\xi + \arctan(l^{-1}\tan\varphi_0)]. \tag{5.2}$$

If $0 < \mu < 1$, then

$$\tan\varphi = j\frac{q\exp(s\xi) - w}{q\exp(s\xi) + w}, \tag{5.3}$$

where

$$j = \sqrt{\frac{\mu}{1-\mu}}, \quad q = 1 + \sqrt{\frac{1-\mu}{\mu}}\tan\varphi_0, \quad s = 2\sqrt{\mu(1-\mu)}, \quad w = 1 - \sqrt{\frac{1-\mu}{\mu}}\tan\varphi_0.$$

For small θ, system (2.5) becomes

$$\begin{gathered}\dot\sigma = -(1-e^2)^{3/2}a_{1c}\frac{R_0^2}{L_0P^2}\cos\rho_0, \quad \rho = \rho_0, \quad L = L_0, \quad \dot\theta = L_0\sin\varphi\cos\varphi\left(\frac{1}{A} - \frac{1}{B}\right), \\ \dot\varphi = L_0\left(\frac{1}{C} - \frac{\sin^2\varphi}{A} - \frac{\cos^2\varphi}{B}\right) - (1-e^2)^{3/2}a_{1c}\frac{R_0^2}{L_0P^2}\left(1 - \tfrac{3}{2}\sin^2\rho_0\right).\end{gathered} \tag{5.4}$$

In these equations, the terms of the order higher than linear in θ are omitted. For small θ, the equation for φ coincides with the corresponding equation for $\theta = 0$, and its solution can be represented in the form (5.1)–(5.3). On integrating Eq. (5.4) for θ with allowance for (5.1), for $\mu > 1$, we obtain

$$\theta^2 = \frac{\theta_0^2}{l^2}(l^2\cos^2\varphi_0 + \sin^2\varphi_0)\left\{\cos^2[r\xi + \arctan(l^{-1}\tan\varphi_0)] + l^2\sin^2[r\xi + \arctan(l^{-1}\tan\varphi_0)]\right\}. \tag{5.5}$$

For $\mu < 0$, the solution of Eq. (5.4) for the nutation angle with allowance for (5.2) has the form

$$\theta^2 = \frac{\theta_0^2 l^2}{l^2\cos^2\varphi_0 + \sin^2\varphi_0}\left\{\cos^2[-r\xi + \arctan(l^{-1}\tan\varphi_0)] + l^2\sin^2[-r\xi + \arctan(l^{-1}\tan\varphi_0)]\right\}^{-1}. \tag{5.6}$$

If $0 < \mu < 1$, then, taking into account (5.3), we obtain

$$\theta = \theta_0\exp\int_0^\xi F(\xi_1)\,d\xi_1, \tag{5.7}$$

where

$$F(\xi) = \frac{j[q^2\exp(2s\xi) - w^2]}{[q\exp(s\xi) + w]^2 + j^2[q\exp(s\xi) - w]^2}.$$

Integrating Eq. (5.4) for σ yields

$$\sigma = \sigma_0 - (1-e^2)^{3/2}a_{1c}\frac{R_0^2}{L_0P^2}t\cos\rho_0. \tag{5.8}$$

Note that for a strongly dynamically symmetric satellite ($A = B + O(\varepsilon^2)$), the integration of Eqs. (2.5) for θ and φ results in the following expressions:

$$\begin{gathered}\theta = \theta_0, \quad L = L_0, \quad \rho = \rho_0, \quad \varphi = \varphi_0 + L_0(\alpha - \gamma)\cos\theta_0 t, \\ \sigma = \sigma_0 - (1-e^2)^{3/2}a_{1c}\frac{R_0^2}{L_0P^2}t\left(1 - \tfrac{3}{2}\sin^2\theta_0\right)\cos\rho_0.\end{gathered} \tag{5.9}$$

Thus, we have investigated the evolution of rotations of a nearly spherically symmetric satellite under the action of the light pressure torque in the approximation taking into account the zeroth and the first harmonics. Some qualitative effects have been demonstrated. It is of considerable interest, both for theory and applications, to investigate this problem for a more complete model of the light pressure torque.

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