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# Optimal Deceleration of Rotations of an Asymmetric Body with a Cavity Filled with Viscous Fluid in a Resistive Medium

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**Abstract**—A minimum-time problem on deceleration of rotations of a free rigid body is studied. It is assumed that the body contains a spherical cavity filled with highly viscous fluid. The body is subjected to a retarding torque of viscous friction. It is assumed that the body is dynamically asymmetric. An optimal control law for the deceleration of rotations of the body is synthesized, and the corresponding time and phase trajectories are determined.

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## INTRODUCTION

Analysis of the motion of hybrid systems (i.e., objects containing elements with distributed and concentrated parameters) is of interest both theoretically and practically. This analysis can be done within the framework of the theory of singularly perturbed problems. Important results were obtained for systems containing “quasi-rigid” bodies. Combined rotational and translational motions of these systems are close (under certain conditions) to the motion of absolutely rigid bodies. The influence of non-ideal features are reduced to the effects of the temporal boundary layer type and to additional perturbing moments in the Euler equations of angular motion of a fictitious rigid body after the completion of transient processes. The analysis of motions of a rigid body with a cavity filled with a viscous fluid in a resistive medium had received much attention [1–6]. The control of rotations of quasi-rigid bodies using concentrated (applied to the frame) torques received less attention. Researchers managed to distinguish a class of systems leading to smooth controls making it possible to apply singular disturbance methods without accumulation of boundary layer type errors appearing in the case of discontinuous (for example, bang-bang) controls [7–9].

In this paper, we investigate the problem of time-optimal deceleration of rotations of a dynamically nonsymmetric body with a spherical cavity filled with highly viscous fluid (for small Reynolds numbers). In addition, the rigid body is subjected to the action of a small retarding torque of linear resistance of the medium. The rotations are controlled by a bounded torque, which can be exerted by vernier jet engines [7]. The model under consideration generalizes the results obtained earlier in [7–12]. The problem of optimal deceleration of rotations of a dynamically symmetric body containing a viscous–elastic element and a cavity filled with fluid is studied in [8]. The problem of time-optimal deceleration of rotations of a dynamically symmetric rigid body with a spherical cavity filled with highly viscous fluid and a moving mass attached to the body by an elastic joint with quadratic dissipation is considered in [9]. The problem of optimal deceleration of rotations of a dynamically symmetric body with a cavity filled with highly viscous fluid is considered in [10], where the rigid body is subjected to a small torque of viscous friction of the external medium. The problem of time-optimal deceleration of rotations of a dynamically asymmetric body in a resistive medium is considered in [11]. Approximate solutions of perturbed problems of time-optimal deceleration of rotations of rigid bodies about the center of mass (including objects with internal degrees of freedom) with applications to the spacecraft and aircraft dynamics were obtained in the monograph [7]. There, the deceleration of bodies having a cavity with viscous fluid was studied. The cases of axisymmetric and asymmetric (in the undisturbed state) bodies with a spherical cavity filled with highly viscous fluid were considered. The deceleration of perturbed rotations of a rigid body close to a spherically symmetric one under the action of the torque exerted by the linear resistance of the medium was analyzed.

## 1. STATEMENT OF THE PROBLEM

We consider a dynamically nonsymmetric rigid body with moments of inertia satisfying, for definiteness, the inequalities  $A_1 > A_2 > A_3$ . Based on the approach described in [7], the equations of controlled rotations projected on the axes of the body-related coordinate system (the Euler equations) can be expressed as [1, 4, 5, 7]

$$J\dot{\boldsymbol{\omega}} + [\boldsymbol{\omega} \times J\boldsymbol{\omega}] = \mathbf{M}^u + \mathbf{M}^r + \mathbf{M}^c. \quad (1.1)$$

Here,  $\boldsymbol{\omega} = (p, q, r)$  is the vector of absolute angular velocity,  $J = \text{diag}(A_1, A_2, A_3)$  is the tensor of body inertia,  $\mathbf{M}^u$  is the vector of control torque,  $\mathbf{M}^r$  is the dissipation torque, and  $\mathbf{M}^c$  is the torque of viscous fluid in the body cavity. The kinetic moment of the body is determined in the standard way as

$$\mathbf{G} = J\boldsymbol{\omega}, \quad \mathbf{G} = (G_1, G_2, G_3), \quad G_1 = A_1 p, \quad G_2 = A_2 q, \quad G_3 = A_3 r,$$

where  $G = |\mathbf{G}| = (G_1^2 + G_2^2 + G_3^2)^{1/2}$  is its magnitude.

To simplify the problem, we introduce structural constraints into system (1.1); in particular, we assume that the feasible values of the control torque  $\mathbf{M}^u$  belong to a sphere [7]). This assumption is not inconsistent with the mass distribution and shape of the rigid body and is often used in attitude control problems. It is also believed that the diagonal tensor of the external resistance torque is proportional to the moment-of-inertia tensor; i.e., the dissipation torque is proportional to the kinetic moment:

$$\mathbf{M}^r = -\lambda J\boldsymbol{\omega}. \quad (1.2)$$

Here,  $\lambda$  is a constant coefficient depending on the medium properties. The resistance acting on the body is represented by a pair of forces. In this case, the projections of the moment of this pair on the major axes of body inertia are  $\lambda A_1 p$ ,  $\lambda A_2 q$ , and  $\lambda A_3 r$  [4, 5]. This assumption is not contradictory.

Next, we assume that the cavity is filled with highly viscous fluid; i.e.,  $\vartheta \gg 1$  ( $\vartheta^{-1} \sim \varepsilon \ll 1$ ). The shape of the cavity is supposed to be almost spherical; then, following [1], for the tensor  $\tilde{P}$  of the viscous forces, we have

$$\tilde{P} = P \text{diag}(1, 1, 1), \quad P = \frac{8\pi\rho a^7}{525\vartheta}, \quad (1.3)$$

where  $\rho$ ,  $\vartheta$  are the fluid density and kinematic viscosity, respectively; and  $a$  is the cavity radius. The tensor  $\tilde{P}$ , which depends only on the cavity shape, characterizes the internal dissipative torque in the quasi-static approximation due to the viscous fluid in the cavity. For simplicity, Eqs. (1.1) use the so-called scalar tensor defined by a single scalar  $P > 0$ . The components of this tensor have the form  $\tilde{P}_{ij} = P\delta_{ij}$ , where  $\delta_{ij}$  are the Kronecker symbols (the tensor  $\tilde{P}$  has this form if the cavity is spherical, for example). If the cavity is significantly nonspherical, there are considerable difficulties in determining the tensor components.

The admissible values of the moment  $\mathbf{M}^u$  of the control forces are assumed to be bounded by the sphere

$$\mathbf{M}^u = b\mathbf{u}, \quad |\mathbf{u}| \leq 1; \quad b = b(t, \mathbf{G}), \quad 0 < b_* \leq b < b^* < \infty, \quad (1.4)$$

where  $b$  is a scalar function bounded in the domain of variation of its arguments  $t$  and  $\mathbf{G}$  according to conditions (1.4). This domain is given a priori or can be estimated from the initial data for  $\mathbf{G}$  ( $\mathbf{G}(t_0) = \mathbf{G}^0$ ) by integrating Eq. (1.1) with respect to  $\boldsymbol{\omega}$ . Below, we suppose that  $b = b(t, \mathbf{G})$  (or  $b = b(t)$  or  $b = \text{const}$ ).

We pose the problem of time-optimal deceleration of rotations

$$\boldsymbol{\omega}(t_0) = \boldsymbol{\omega}^0, \quad \boldsymbol{\omega}(T) = 0, \quad T \rightarrow \min_{\mathbf{u}}, \quad |\mathbf{u}| \leq 1. \quad (1.5)$$

It is required to find an optimal control  $u = u(t, \boldsymbol{\omega})$ , the corresponding trajectory  $\boldsymbol{\omega}(t, t_0, \boldsymbol{\omega}^0)$ , the time  $T = T(t_0, \boldsymbol{\omega}^0)$ , and the Bellman function  $W = T(t, \boldsymbol{\omega})$ . Based on dynamic programming and the Schwarz inequality, under the simplifying condition on the coefficient  $b$  ( $b(t, \mathbf{G}) = b_0(t, G)$ , where the zero subscript will be omitted below) a time-optimal control is constructed in the form (see [7])

$$M_p = -b \frac{A_1 p}{G}, \quad M_q = -b \frac{A_2 q}{G}, \quad M_r = -b \frac{A_3 r}{G}, \quad b = b(t, G). \quad (1.6)$$

With regard to external force factors, the torque of viscous fluid in the cavity  $\mathbf{M}^c$  is determined as (see [1])

$$\mathbf{M}^c = \frac{P\rho}{\nu} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}, \quad (1.7)$$

where

$$m_1 = p \left( \lambda^2 + \frac{b^2}{G^2} \right) + \frac{1}{A_1} \left( \lambda + \frac{b}{G} \right) \left( 3qr(A_3 - A_2) + \frac{G\alpha_{33}}{1 - \alpha_{33}^2} q(\alpha_{31}^2 + \alpha_{32}^2) - Gr\alpha_{32} \right) + \frac{2\lambda b}{G} p + \frac{P}{A_1 A_2 A_3} [q^2 A_2 (A_1 - A_2)(A_2 - A_3 + A_1) + r^2 A_3 (A_1 - A_3)(A_3 - A_2 + A_1)].$$

The expressions for  $m_2$  and  $m_3$  are obtained from  $m_1$  in (1.7) by a cyclic permutation of  $A_1, A_2, A_3$  and  $p, q, r$ . The coefficients  $\lambda^2 + \frac{b^2}{G^2}$ ,  $\lambda + \frac{b}{G}$ , and  $\frac{2\lambda b}{G}$  in  $m_i, i = 1, 2, 3$  remain unchanged, and the terms containing  $\alpha_{31}, \alpha_{32}, \alpha_{31}^2 + \alpha_{32}^2$  have a similar form. The direction cosines  $\alpha_{ij}$  are expressed in terms of the Euler angles  $\varphi, \psi$ , and  $\theta$  according to well-known formulas [12]. Neglecting the influence of  $\mathbf{M}^u$  and  $\mathbf{M}^r$  on  $\mathbf{M}^c$ , we obtain the torque of viscous fluid in the cavity in the form

$$\mathbf{M}^c = \frac{P}{A_1 A_2 A_3} \begin{pmatrix} p[q^2 A_2 (A_1 - A_2)(A_2 - A_3 + A_1) + r^2 A_3 (A_1 - A_3)(A_3 - A_2 + A_1)] \\ q[r^2 A_3 (A_2 - A_3)(A_3 - A_1 + A_2) + p^2 A_1 (A_2 - A_1)(A_1 - A_3 + A_2)] \\ r[p^2 A_1 (A_3 - A_1)(A_1 - A_2 + A_3) + q^2 A_2 (A_3 - A_2)(A_2 - A_1 + A_3)] \end{pmatrix}. \quad (1.8)$$

accurate to a first-order infinitesimal  $\varepsilon$ .

We consider this expression only in the first approximation., The equations of controlled motion (1.1) simplified on the basis of expression (1.8) in projections on the major central axes of inertia have the form

$$\begin{aligned} A_1 \dot{p} + (A_3 - A_2)qr &= -b \frac{A_1 p}{G} - \lambda A_1 p \\ &+ \frac{P}{A_1 A_2 A_3} p [q^2 A_2 (A_1 - A_2)(A_2 - A_3 + A_1) + r^2 A_3 (A_1 - A_3)(A_3 - A_2 + A_1)], \\ A_2 \dot{q} + (A_1 - A_3)pr &= -b \frac{A_2 q}{G} - \lambda A_2 q \\ &+ \frac{P}{A_1 A_2 A_3} q [r^2 A_3 (A_2 - A_3)(A_3 - A_1 + A_2) + p^2 A_1 (A_2 - A_1)(A_1 - A_3 + A_2)], \\ A_3 \dot{r} + (A_2 - A_1)pq &= -b \frac{A_3 r}{G} - \lambda A_3 r \\ &+ \frac{P}{A_1 A_2 A_3} r [p^2 A_1 (A_3 - A_1)(A_1 - A_2 + A_3) + q^2 A_2 (A_3 - A_2)(A_2 - A_1 + A_3)]. \end{aligned} \quad (1.9)$$

The kinematic relations are omitted because Eqs. (1.9) form a closed system. These equations are further analyzed. Let us make the statement of problem (1.4) more precise.

## 2. SOLUTION OF THE OPTIMAL DECELERATION PROBLEM

Let us note that the torque exerted by viscous fluid in the cavity is internal, while the torque of the linear drag of the medium is external. Multiplying the first equation in (1.8) by  $G_1$ , the second equation by  $G_2$ , and the third equation by  $G_3$ , and summing them (scalar product  $\mathbf{G} \cdot \mathbf{G}$ ), we obtain a scalar equation to be integrated:

$$\dot{G} = -b(t, G) - \lambda G, \quad G(t_0) = G^0. \quad (2.1)$$

Upon solving Cauchy problem (2.1), we obtain from the condition of stopping the rotation (1.5) the required expression for the time  $T = T(t_0, G^0)$  and the Bellman function  $T(t, G)$ . Recall that  $\mathbf{G} = J\boldsymbol{\omega}$ .

In the general case, for an arbitrary function  $b = b(t, G)$  in (2.1), the analytical integration of the Cauchy problem is complicated; however, it can be solved numerically. Equations (2.1) imply that the evolution of the magnitude of the kinetic moment  $G$  is affected by the control moment and the medium drag. The internal torque of the viscous fluid in the cavity has no effect. If  $b = b(t)$  (i.e., the function  $b(t)$  is independent of  $\mathbf{G}$ ), we obtain the solution of boundary problem (2.1)

$$G(t) = G^0 \exp(-\lambda(t - t_0)) - \int_{t_0}^t b(\tau) \exp(-\lambda(t - \tau)) d\tau, \quad (2.2)$$

where

$$G^0 = \exp(-\lambda t_0) \int_{t_0}^T b(\tau) \exp(\lambda \tau) d\tau.$$

According to (1.4), Eq. (2.2) is solvable with respect to the unknown  $T$ , which leads to the construction of the time-optimal solution. Here,  $t$  is the current time of deceleration and  $T$  is the optimal time. For  $b = \text{const}$  and  $t_0 = 0$ , the solutions of equation (2.1) and boundary problem (2.2) are written as

$$G(t) = \frac{1}{\lambda} [(G^0 \lambda + b) \exp(-\lambda t) - b], \quad T = \frac{1}{\lambda} \ln \left( G^0 \frac{\lambda}{b} + 1 \right). \quad (2.3)$$

Next, we consider in detail case (2.3). Let us multiply the first equation in (1.8) by  $p$ , the second equation by  $q$ , the third equation by  $r$ , and sum the results. The resulting expression for the derivative of the kinetic energy  $H$  is

$$\begin{aligned} \dot{H} = & -\frac{2bH}{G^2} - 2\lambda H + \frac{P}{A_1 A_2 A_3} [p^2 q^2 (A_1 - A_2)^2 (A_3 - A_1 - A_2) \\ & + p^2 r^2 (A_1 - A_3)^2 (A_2 - A_1 - A_3) + q^2 r^2 (A_2 - A_3)^2 (A_1 - A_2 - A_3)]. \end{aligned} \quad (2.4)$$

Consider an undisturbed motion ( $b = \lambda = \varepsilon = 0$ ). Recall that the cavity contains a highly viscous fluid and  $\vartheta^{-1} \sim \varepsilon \ll 1$ , where  $\vartheta$  is the kinematic viscosity. In the absence of perturbations, the rotation of the rigid body is a Euler–Poinsot motion. The variables  $G$  and  $H$  become constant and  $\varphi$ ,  $\psi$ , and  $\theta$  are functions of time  $t$ . The slow variables in the perturbed motion are  $G$  and  $H$ , and the fast variables are the Euler angles  $\varphi$ ,  $\psi$ , and  $\theta$ .

Consider a motion under the condition  $2HA_1 \geq G^2 > 2HA_2$  corresponding to the trajectories of the kinetic moment vector, which envelope the major torque axis  $Oz_1$ . Define

$$k^2 = \frac{(A_2 - A_3)(2HA_1 - G^2)}{(A_1 - A_2)(G^2 - 2HA_3)} \quad (0 \leq k^2 \leq 1), \quad (2.5)$$

which is the module of elliptic functions describing this motion and is a function of the kinetic moment  $G$  and the kinetic energy  $H$  (in the case of unperturbed motion, it is a constant).

To construct the averaged first-approximation system of equations, we substitute the solution of the unperturbed Euler–Poinsot motion into the right-hand side of Eq. (2.4) and average over the variable  $\psi$  and then over time  $t$  taking into account the dependences of  $\varphi$  and  $\theta$  on  $t$ . Here, we retain the notation for the slow variables  $G$  and  $H$ . As a result, we obtain

$$\begin{aligned} \frac{dH}{dt} = & -\frac{2bH}{G} - 2\lambda H - \frac{4PH^2(A_1 - A_3)(A_1 - A_2)(A_2 - A_3)}{3A_1^2 A_2^2 A_3^2 S^2(k)} \\ & \times \{A_2(A_1 - A_3)(A_1 + A_3 - A_2)[k^2 V(k) - U(k)] + A_1(A_2 - A_3)(A_3 + A_2 - A_1)[(k^2 - 2)U(k) + k^2] \\ & + A_3(A_1 - A_2)(A_1 + A_2 - A_3)[(1 - 2k^2)U(k) + k^2]\}, \end{aligned} \quad (2.6)$$

where

$$U(k) = 1 - \frac{E(k)}{K(k)}, \quad S^2(k) = [A_2 - A_3 + (A_1 - A_2)k^2]^2, \quad V(k) = 1 + \frac{E(k)}{K(k)}.$$

Here,  $K(k)$  and  $E(k)$  are the complete elliptic integrals of the first and second kind, respectively [13]. Equation (2.6) implies that the resistance of the medium and the torque of the viscous fluid in the body cavity as well the control moment cause the evolution of the kinetic energy  $H$  of the body. The expression in the braces on the right-hand side of equation (2.6) is positive (for  $A_1 > A_2 > A_3$ ) because of the inequalities  $(1 - k^2)K \leq E \leq K$  (see [13]). Consequently,  $dH/dt < 0$  because  $H > 0$ ; i.e.,  $H$  is a strictly decreasing variable for any  $k^2 \in [0, 1]$ . Note that Eq. (2.6) at has an essential singularity as  $G \rightarrow 0$ .

Differentiating expression (2.5) for  $k^2$  with regard to (2.6), we obtain a the differential equation

$$\frac{dk^2}{dt} = \frac{PG^2(A_1 - A_3)[A_2(A_1 + A_3 - A_2) + 2A_1A_3]}{3A_1^2A_2^2A_3^2} \left\{ (1 - \chi)(1 - k^2) - [(1 - \chi) + (1 + \chi)k^2] \frac{E(k)}{K(k)} \right\}, \quad (2.7)$$

where

$$\chi = \frac{3A_2[(A_1^2 + A_3^2) - A_2(A_1 + A_3)]}{(A_1 - A_3)[A_2(A_1 + A_3 - A_2) + 2A_1A_3]}.$$

Equations (2.6) and (2.7) were obtained by the method of averaging [1, 2, 8, 10]. This corresponds to the fact that the kinetic energy of the body rotation is much greater than the control vector magnitude, the resistance of the medium is assumed to be weak of the infinitesimal order  $\varepsilon$ , and the cavity is filled with highly viscous fluid.

The value  $k^2 = 1$  is associated with the equality  $2HA_2 = G^2$ , which corresponds to the separatrix of the Euler–Poinsoot motion. Equation (2.7) describes the averaged motion of the endpoint of the kinetic moment vector  $\mathbf{G}$  on a sphere of radius  $G$ . Notice that the evolution of  $k^2$  is affected only by the torque of the viscous fluid in the cavity, and, because this equation is integrated independently, the influence of the torque of the viscous fluid in the cavity, the control moment, and the resistance moment is partially separated. An analysis of Eq. (2.7) shows that there are no stationary values of  $k$  except for  $k = 0$  and 1.

### 3. NUMERICAL CALCULATION

We reduce Eq. (2.6) and (2.7) and the differential equation for the kinetic moment for  $b = \text{const}$  to a dimensionless form. As the characteristic parameters of the problem, we use the value of the kinetic moment at the initial time  $G_0 = G(t_0)$  and the time  $T$  (2.3):

$$\tilde{G} = \frac{G}{G_0}, \quad \tilde{t} = \frac{t}{T}.$$

The value of the dimensionless kinetic energy is defined (see [1]) as

$$\tilde{H} = \frac{2HA_1}{G_0^2}.$$

We obtain a dimensionless system in the form

$$\begin{aligned} \frac{d\tilde{G}}{d\tilde{t}} &= - \left( \frac{b}{G_0} + \lambda\tilde{G} \right) T, \\ \frac{dk^2}{d\tilde{t}} &= \frac{PT\tilde{G}^2 G_0^2 (A_1 - A_3)[A_2(A_1 + A_3 - A_2) + 2A_1A_3]}{3A_1^3A_2^2A_3^2} \\ &\times \left\{ (1 - \chi)(1 - k^2) - [(1 - \chi) + (1 + \chi)k^2] \frac{E(k)}{K(k)} \right\}, \end{aligned}$$

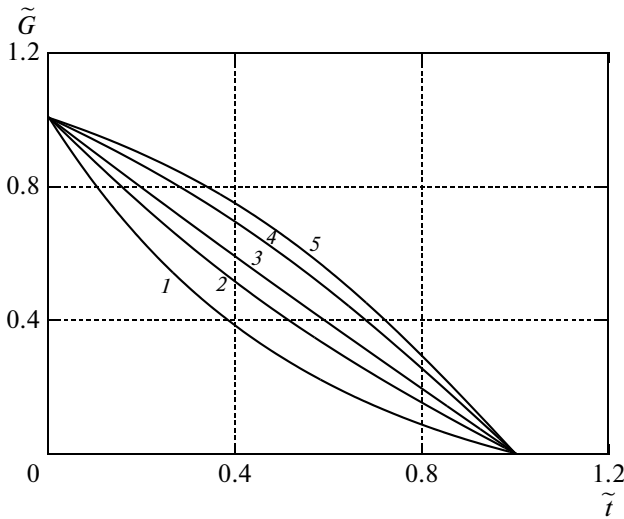


Fig. 1.

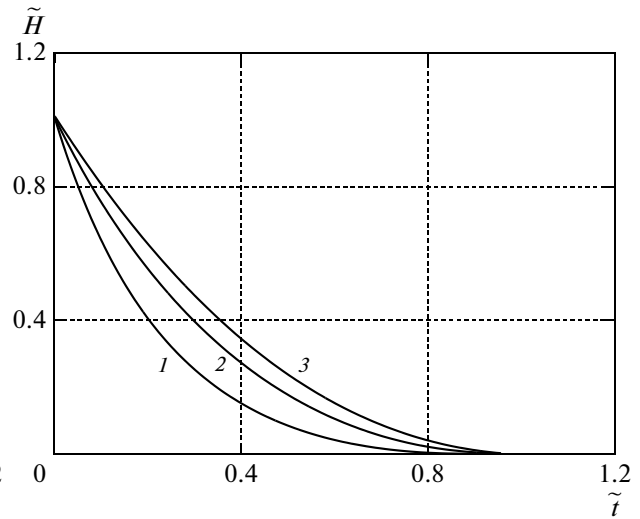


Fig. 2.

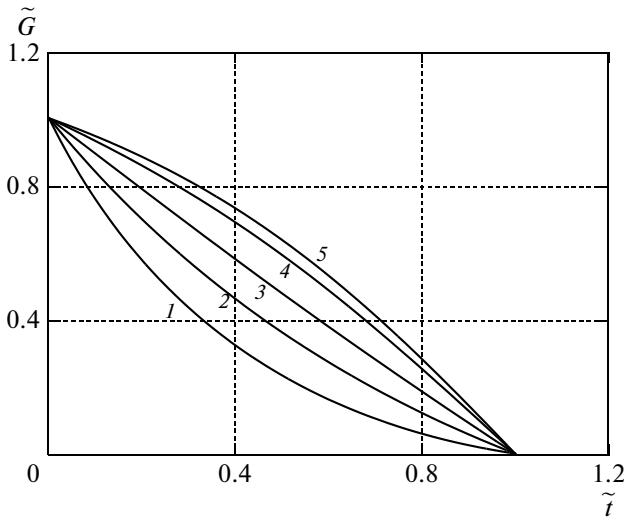


Fig. 3.

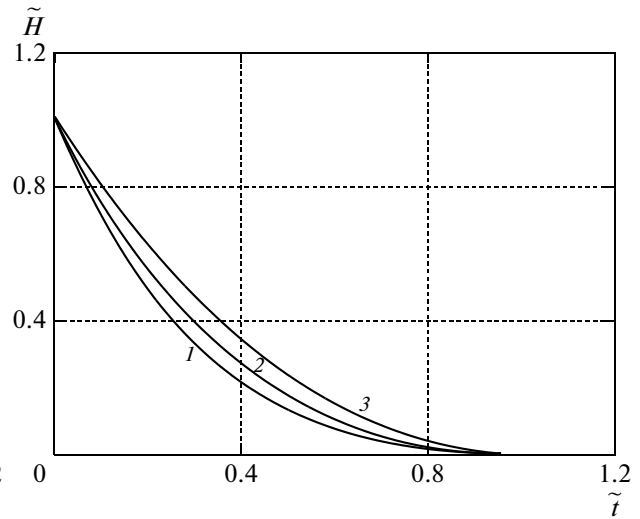


Fig. 4.

$$\frac{d\tilde{H}}{d\tilde{t}} = -T \left( \frac{2b\tilde{H}}{\tilde{G}G_0} + 2\lambda\tilde{H} + \frac{4PG_0^2\tilde{H}^2(A_1 - A_3)(A_1 - A_2)(A_2 - A_3)}{3A_1^3A_2^2A_3^2S^2(k)} \right) \times \{A_2(A_1 - A_3)(A_1 + A_3 - A_2)[k^2V(k) - U(k)] + A_1(A_2 - A_3)(A_3 + A_2 - A_1)[(k^2 - 2)U(k) + k^2] + A_3(A_1 - A_2)(A_1 + A_2 - A_3)[(1 - 2k^2)U(k) + k^2]\} \quad (3.1)$$

Here, we performed averaging because expressions (2.4) and (2.5) imply that  $H$  and  $k^2$  are slow variables. We make a numerical integration over the interval  $[0, 1]$ , which corresponds to complete body deceleration. The initial function values for this calculation were  $\tilde{G}(0) = G_0 = 1$ ,  $\tilde{H}(0) = 1$ , and  $k^2(0) \approx 1$ . The moments of inertia have the values (see [1]):  $A_1 = 8$ ,  $A_2 = 6$ , and  $A_3 = 4$ . The calculations were performed for various values of  $\lambda$ ,  $b$ , and  $P$ , which makes it possible to study the influence of different force factors on the character of the rigid body deceleration. For each case, we first calculated the deceleration time and then the characteristics of the body motion in the corresponding time interval.

Figures 1 and 2 illustrate the numerical analysis for  $P = 10^{-1}$ ,  $b = 10^{-1}$ , and  $\lambda = 0.5, 10^{-1}, 10^{-2}$  (curves 1, 2, and 3, respectively). It is seen that the decrease in the moment of medium resistance forces leads to a

decreased gradient of the body deceleration and an almost linearly dependence of the kinetic moment (curve 3 in Fig. 1).

Figures 3 and 4 show the results of calculations for  $P = 10^{-1}$ ,  $\lambda = 10^{-1}$ , and  $b = 10^{-2}$ ,  $5 \times 10^{-2}$ ,  $5 \times 10^{-1}$  (curves 1, 2, and 3, respectively). It is seen that the increase in the moment of control forces (curve 3 in Fig. 4) leads to a speedup of the body deceleration and an almost linearly changing magnitude of the kinetic moment at large values of  $b$  (curve 3 in Fig. 3).

The change of  $P$  from 1 to  $10^{-2}$  has no effect on the general behavior of the functions  $\tilde{G} = \tilde{G}(\tilde{t})$  and  $\tilde{H} = \tilde{H}(\tilde{t})$  because the torque of the viscous fluid in the cavity does not appear in the first equation of system (3.1) and its effect on the change of the kinetic energy is smaller than the influence of the moment of resistance forces and the control moment. The numerical results show that, for the values of  $\lambda$ ,  $b$ , and  $P$  indicated above, the module of elliptic functions  $k^2$  insignificantly decreases from around 1 to 0.9996.

#### 4. INVESTIGATION OF THE RIGID BODY DECELERATION AS $k^2 \rightarrow 1$

Consider (2.7) as  $k^2 \rightarrow 1$ , which corresponds to trajectories of the kinetic moment vector near the separatrix. According to [13], the elliptic integrals can be expanded into series having regard to small  $k'$  values of the second order (complementary modulus of elliptic functions), where

$$k' = \sqrt{1 - k^2}, \quad \text{i.e.,} \quad k' \ll 1. \tag{4.1}$$

Using the terms of the first and second infinitesimal terms of  $k'$ , one can find the asymptotics of the solution of differential equation (2.7):

$$t = \frac{(1 - k^2)}{2cG^2} \left( \ln \frac{4}{\sqrt{1 - k^2}} + \frac{1}{2} \right), \tag{4.2}$$

where

$$c = \frac{P(A_1 - A_3)[A_2(A_1 + A_3 - A_2) + 2A_1A_3]}{3A_1^2A_2^2A_3^2}.$$

For  $b = \text{const}$ , we have

$$t = \frac{(1 - k^2)}{c[(G^0\lambda + b)\exp(-\lambda t) - b]^2} \left( \ln \frac{4}{\sqrt{1 - k^2}} + \frac{1}{2} \right). \tag{4.3}$$

Equation (4.3) implicitly specifies the dependence  $k^2 = k^2(t)$ . According to the numerical calculation, the elliptic functions modulus  $k^2 \rightarrow 1$  also decreases from around 1 to 0.9996 as in Section 3. If  $k^2 \rightarrow 1$ , Eq. (2.6) with account for the first-order infinitesimals  $k'$  takes the form

$$\frac{dH}{dt} = -2 \left( \frac{b}{G} + \lambda \right) H + \frac{2\alpha H^2(1 - k^2)(A_2^2 - A_1A_2 - 2A_1A_3 - A_2A_3)}{(A_1 - A_3)(4cG^2t - 1 + k^2)}, \tag{4.4}$$

where

$$\alpha = \frac{4P(A_1 - A_3)(A_1 - A_2)(A_2 - A_3)}{3A_1^2A_2^2A_3^2}.$$

As in Section 3, we reduce Eq. (4.4) to a dimensionless form

$$\frac{d\tilde{H}}{d\tilde{t}} = -2 \left( \frac{bG_0}{\tilde{G}} + \lambda \right) \tilde{H}T + \frac{\alpha \tilde{H}^2 G_0^2 (1 - k^2) (A_2^2 - A_1A_2 - 2A_1A_3 - A_2A_3)}{A_1(A_1 - A_3)(4c\tilde{G}^2 G_0^2 T\tilde{t} - 1 + k^2)}. \tag{4.5}$$

Equation (4.5) was numerically integrated taking into account the two equations in system (3.1) for  $\lambda$ ,  $b$ , and  $P$  equal to 0.1 and the initial values of the functions  $\tilde{G}(0) = G_0 = 1$ ,  $\tilde{H}(0) = 1$ , and  $k^2(0) \approx 1$ . For the moments of inertia, the following values are specified (see [1]):  $A_1 = 8$ ,  $A_2 = 6$ , and  $A_3 = 4$ . The numerical integration interval was  $[0, 1]$ . The results of the numerical analysis are illustrated in Fig. 5 (curve 4); it is seen that the kinetic energy has the same shape as in the calculation described in Section 3.



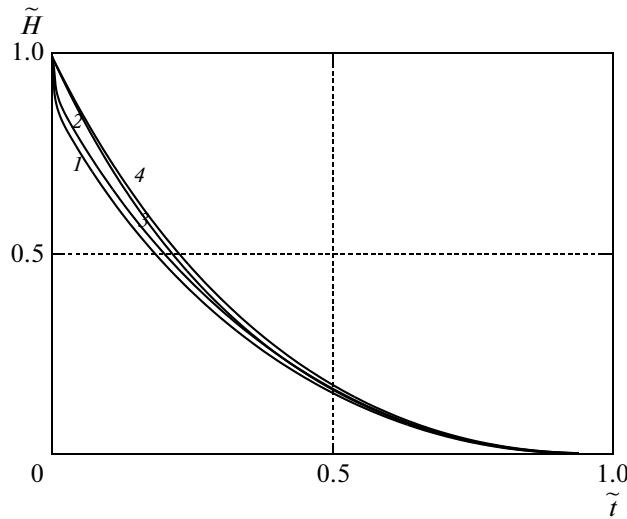


Fig. 5.

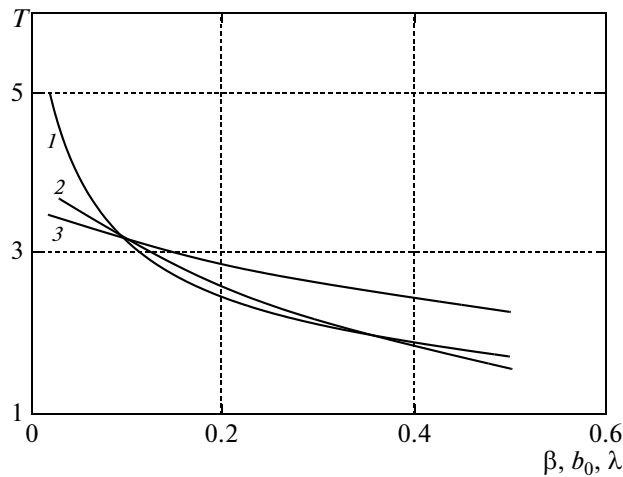


Fig. 6.

## 5. SOLUTION OF THE OPTIMAL DECELERATION PROBLEM UNDER THE ASSUMPTION $b = b_0 + \beta t$

The deceleration time of a rigid body can be determined from Eq. (2.1); it depends on the coefficients  $\beta$ ,  $b_0$ , and  $\lambda$  characterizing the control moment and the moment of resistance forces, respectively. The numerical integration illustrates how the deceleration time depends on these parameters (see Fig. 6). Curves 1, 2, 3 illustrate the dependence of the deceleration time on the parameters  $\beta$ ,  $b_0$ , and  $\lambda$ , respectively. For each curve, the deceleration time was calculated in the range from 0.01 to 0.5 for the corresponding parameter (the values of the other parameters were taken to be 0.1). It can be seen that the deceleration time for all the curves is a minimized by the maximum values of the parameters in the admissible range. Curves 2 and 3 are close to straight lines, and curve 1 is close to an exponential curve. The minimum deceleration time of the rigid body was obtained for the parameter  $b_0$ , which characterizes the value of the control moment at the initial time.

System of equations (3.1) was numerically integrated for different values of parameters  $P$ ,  $\lambda$ ,  $b_0$ , and  $\beta$  with account for the law  $b = b_0 + \beta t$ . Figure 3 (curves 4 and 5) corresponds to the numerical calculation for constant parameters of resistance torque and viscous fluid torque in the cavity  $P = 0.1$  and  $\lambda = 0.1$  for

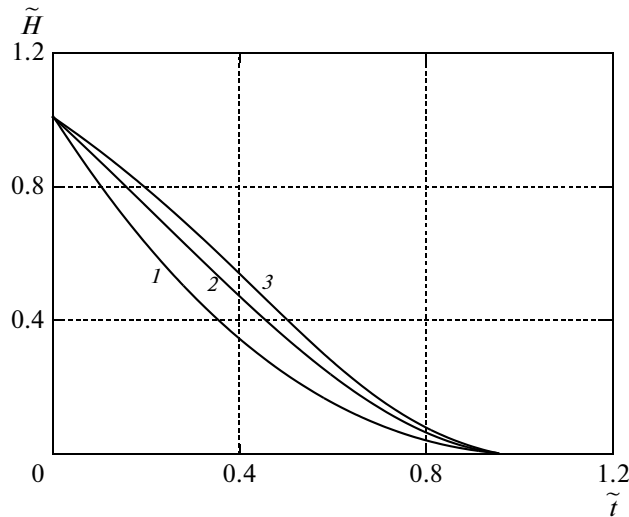


Fig. 7.

different values of the control moment:  $b_0 = 0.1$  and  $\beta = 0.1$  (curve 4), and  $b_0 = 0.01$  and  $\beta = 0.1$  (curve 5). Under this law of change in the control moment, the behavior of the function  $\tilde{G} = \tilde{G}(\tilde{t})$  is substantially different from that of the kinetic moment function at  $b = \text{const}$ .

We performed the numerical integration for constant parameters of the control moment under the law  $b = b_0 + \beta t$  at different values of the resistance torque. For  $P = 0.1$ ,  $\beta = 0.1$ , and  $b_0 = 0.1$ , the numerical results are shown in Fig. 1, where  $\lambda = 0.1$  (curve 4) and  $\lambda = 0.01$  (curve 5). It is seen that the higher the coefficient of the resistance torque, the faster is the body deceleration. The decrease pattern of the kinetic moment differs from the function body shown in Fig. 3. The numerical analysis makes it possible to conclude that, for  $b = \text{const}$ , the body deceleration is faster in the first half of the dimensionless time  $\tilde{t}$ , and, in the case  $b = b_0 + \beta t$ , it is faster in the second half.

Figure 7 shows the results of numerical calculation of the rigid body kinetic energy for  $b = b_0 + \beta t$ . We used the following values of the perturbing moment parameters:  $P = 0.1$  and  $\lambda = 0.1$ . The parameters of the control moment are:  $b_0 = 0.01$  and  $\beta = 0.01$  (curve 1);  $b_0 = 0.1$  and  $\beta = 0.1$  (curve 2); and  $b_0 = 0.01$  and  $\beta = 0.1$  (curve 3). The behavior of the kinetic energy is similar to that shown in Fig. 2. Here, the function  $\tilde{H} = \tilde{H}(\tilde{t})$  at  $b = b_0 + \beta t$  decreases with lower gradients.

## 6. NUMERICAL INVESTIGATION OF RIGID BODY DECELERATION FOR DIFFERENT INITIAL VALUES OF $k^2$

The optimal control problem for a nonsymmetric rigid body was studied in Section 3.5 for the initial value of the elliptic functions modulus  $k^2(0) \approx 1$ . We analyze the behavior of the kinetic moment and kinetic energy for different initial values of  $k^2$  for rigid bodies with various mass geometry.

Consider a rigid body with the moments of inertia  $A_1 = 8$ ,  $A_2 = 6$ ,  $A_3 = 4$ , and specify the coefficients of the perturbing moments as  $P = 10^{-1}$ ,  $b = 10^{-1}$ ,  $\lambda = 10^{-1}$ . For different initial values of the modulus  $k^2 = 0.1, 0.5, \approx 1$ , we obtained the plots of changes in the kinetic energy of the rigid body (see curve 4 in Fig. 5). The curves are almost identical—the difference between the functions  $\tilde{H} = \tilde{H}(\tilde{t})$  is only at the fifth decimal digit.

Consider a rigid body with the moments of inertia  $A_1 = 1.5$ ,  $A_2 = 1$ ,  $A_3 = 0.333$ , for the same coefficients of perturbing moments for various initial values of the elliptic functions modulus  $k^2 = 0.1, 0.5, \approx 1$ . Figure 5 (curve 3) shows the results of numerical calculation as a single curve because the calculation results are slightly different only in the third decimal digit.

Consider the numerical results obtained for the rigid body with the mass geometry  $A_1 = 0.1875$ ,  $A_2 = 0.16$ ,  $A_3 = 0.0938$ . They are shown in Fig. 5 (curves 1 and 2). It is seen that the curves differ in the beginning of the process and then converge to same value when the body deceleration completes. Curves 1 and 2 correspond to the initial values of the elliptic functions modulus  $k^2 = 0.9999$  and 0.5.

## 7. INVESTIGATION OF THE MOTION OF RIGID BODY FOR $k^2 \ll 1$

- 2.3 Consider the body motion for small values of  $k^2 \ll 1$ , which corresponds to the motions that are close to rotations about the axis  $Oz_1$ . In this case, the right-hand side of the second equation in (3.1) can be simplified using expansions of the complete elliptic integrals into series powers of  $k^2$  (see [13]):

$$\frac{dk^2}{d\tilde{t}} = -\frac{TPG_0^2(A_1 - A_3)(3 + \chi)[A_2(A_1 + A_3 - A_2) + 2A_1A_3]}{6A_1^2A_2^2A_3^2}k^2\tilde{G}^2. \quad (7.1)$$

The quantity  $\chi$  is determined from (2.7). If  $b = \text{const}$ , the law of change in the kinetic moment has the dimensionless form

$$\tilde{G}(\tilde{t}) = \left( \tilde{G}_0 + \frac{b}{\tilde{G}_0\lambda} \right) \exp(-\lambda T\tilde{t}) - \frac{b}{\tilde{G}_0\lambda}. \quad (7.2)$$

Substituting (7.2) into (7.1), we obtain a differential equation that has the analytical solution

$$k^2 = k_0^2 \exp \left( \frac{TPG_0^2(A_1 - A_3)(3 + \chi)[A_2(A_1 + A_3 - A_2) + 2A_1A_3]}{6A_1^2A_2^2A_3^2} \right) \times \left[ \frac{(G_0^2\lambda + b)^2}{2T\lambda^3G_0^2} (\exp(-2T\lambda\tilde{t}) - 1) + \frac{2(G_0^2\lambda + b)b}{T\lambda^3G_0^2} (1 - \exp(-T\lambda\tilde{t})) - \frac{b^2\tilde{t}}{G_0^2\lambda^2} \right]. \quad (7.3)$$

As in the case of arbitrary elliptic functions modulus  $k^2$ , function (7.3) is decreasing, but the decrease is insignificant of about  $10^{-3}$ .

## CONCLUSIONS

The problem of time-optimal deceleration of rotations of a dynamically nonsymmetric quasi-rigid body in a resistive medium was studied analytically and numerically. The asymptotic approach made is possible to determine the control, time (Bellman's function), evolutions of the magnitude of the elliptic functions modulus  $k^2$ , and dimensionless kinetic energy and kinetic moment. The qualitative properties of the optimal motion were found.

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