УДК -72.02

USE OF INTEGRAL CALCULUS FOR BUILDING DEVELOPMENTS OF UNDEVELOPABLE SURFACES OF REVOLUTION

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Abstract. Generally, students of architecture use rather simple geometric elements: planes, prisms, cylinders, etc to represent their architectural decisions with paper and card models. However, nowadays modern architects are trying to diversify the architectural environment with more complex surfaces. So, the students of architecture also try to complicate their paper models with curvilinear forms. To facilitate students' architectural modeling and to render more precise building of paper-made curvilinear surfaces, the method of constructing development of elevation view surfaces using integral calculation has been proposed. These developments are constructed similar to the sphere development, i.e. every given surface is divided into 12 identical segments. The task is to find the full length of the main meridian and the arcs of parallels which make part of the constructed segment. The main meridian length is defined by the integral arc length formula.

Keywords: development, parabola, hyperbola, chain line, tractrix, catenoid, pseudosphere.

ВИКОРИСТАННЯ ІНТЕГРАЛЬНОГО ЧИСЛЕННЯ ДЛЯ ПОБУДОВИ РОЗГОРТОК ПОВЕРХОНЬ ОБЕРТАННЯ

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Анотація. Для виразу своїх архітектурних рішень в моделях з паперу та картону студенти використовують достатньо прості геометричні елементи – площини, призми, циліндри, тощо. Але на даний час сучасні архітектори намагаються різноманітити архітектурне середовище більш складними поверхнями. Відповідно до цього, студенти архітектурного інституту також намагаються ускладнювати свої паперові моделі криволінійними формами. Для полегшення студентської роботи в архітектурному макетуванні і більш точного конструювання паперових криволінійних поверхонь розроблено метод побудови розгорток поверхонь обертання з використанням інтегрального числення. Розгортки поверхонь обертання побудовані за аналогією розгортки сфери, тобто кожна задана поверхня розбивається на 12 однакових сегментів. Задача зводиться до відшукання повної довжин головного меридіану і дуг паралелей, які входять до побудованого сегменту. Довжину головного меридіану визначають за ін-

тегральною формулою довжини дуги.

Ключові слова: умовна розгортка, парабола, гіпербола, ланцюгова лінія, трактриса, катеноїд, псевдосфера.

ИСПОЛЬЗОВАНИЕ ИНТЕГРАЛЬНОГО ИСЧИСЛЕНИЯ ДЛЯ ПОСТРОЕНИЯ РАЗВЕРТОК ПОВЕРХНОСТЕЙ ВРАЩЕНИЯ

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Аннотация. Для выражения своих архитектурных решений в моделях з бумаги и картона студенты используют достаточно простые геометрические элементы – плоскости, призмы и цилиндры. Однако, в настоящее время современные архитекторы стараются разнообразить архитектурную среду более сложными поверхностями. Соответственно этому, студенты архитектурного института также стараются усложнять свои бумажные модели криволинейными формами. Для облегчения студенческой работы в архитектурном макетировании и более точного конструирования криволинейных поверхностей разработан метод построения разверток поверхностей вращения с использованием интегрального исчисления. Развертки поверхностей вращения построены аналогично развертке сферы, то есть каждая заданная поверхность разбивается на 12 одинаковых сегментов. Задача сводится к отысканию полной длины главного меридиана и дуг параллелей, которые принадлежат построенному сегменту Длину главного меридиана определяют при помощи интегральной формулы длины дуги.

Ключевые слова: условная развертка, парабола, гипербола, цепная линия, трактриса, катеноид, псевдосфера.

Introduction. The descriptive geometry and drawing course includes «Developments of surfaces». At the practical lessons the tasks with developments of pyramids, prisms, rotating cone and rotating cylinder are necessarily performed. And Architectural Faculty students also solve tasks of sphere developments and rotational surfaces, using the method of cylinders or cones that circle or are circles inside or outside the given surfaces.

Analysis of recent research and publications. Such developments are called conditional or approximate developments [1, 2], since the curves are replaced by straight lines. Of course, all the drawings are done on the paper with pencil and drawing tools.

Purpose and tasks. To increase the developments' graphic accuracy, we propose to use differential calculations to define the lengths of arcs, replaced by segments in the drawing. We will construct the developments of rotational surfaces similar to the sphere development [2, 3], that divides every given surface into 12 identical segments. The task will be reduced to find the full lengths of the main meridian and the arcs of the parallels which make part of the constructed segment.

Presenting main material. From the integral calculations studies, it is known that the arc length of the curve given conventionally as $y = f(x)$ is calculated with the formula [4]:

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx. \quad (1)$$

Since we have 12 segments, the arc length for a parallel having radius R is determined with $\frac{2\pi R}{12}$ formula. To simplify the drawing construction, we'll calculate only half-length of the arc in the segment: $a_n = \frac{2\pi R_n}{24}$. (2)

The surface developments construction stages are the following:

1. Partition of the $y = f(x)$ curve representing the surface main meridian into several parts.
2. Calculating these arcs lengths L_n with the formula (1).
3. Calculating the surface parallels radii R_n and the arcs lengths a_n with the formula (2).
4. Drawings with AutoCAD graphic editor a half or a part of the involute segment.
5. Building, with the use of graphic editor commands «Mirror» and «Copy», a full involute of the given surface.

Considered surfaces are the following: rotational paraboloid, hyperboloid of one sheet, hyperboloid of two sheets, catenoid and pseudosphere. To render easer computation for the main meridians curves, we take the simplest functions: $y = x^2$, $y = \frac{1}{x}$, $y = \frac{1}{2}(e^x + e^{-x})$.

Building a development of a rotational paraboloid. For example, we give a rotational paraboloid which generatrix equation is $y = x^2$. The equation of the surface becomes: $x^2 + y^2 = z$.

Now we consider the curve $y = x^2$ on the plane *xOy*, divided into several arcs (Fig. 1). We have segments of integration: $x \in$ [0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2].

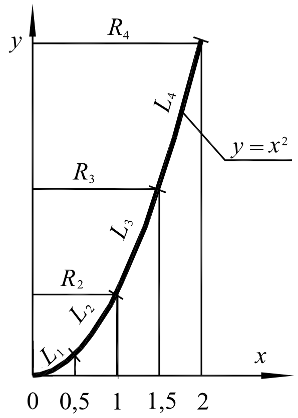

Fig. 1. Generatrix of a paraboloid on the *xOy* axes

With the formula (1) we work out an integral for calculating the parabola arc length:

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx = \int_a^b \sqrt{1 + ((x^2)')^2}\,dx = \int_a^b \sqrt{1 + 4x^2}\,dx.$$

This integral calculation is rather cumbersome (by-parts integration to be applied) [4], therefore, we write only the end result:

$$L = \int_a^b \sqrt{1 + 4x^2}\,dx = \left(\frac{1}{2}x\sqrt{1 + 4x^2} + \frac{1}{4}\ln\left|2x + \sqrt{1 + 4x^2}\right|\right)\Big|_a^b.$$

Now we substitute the integration boundary into the final formula:

$$L_1 = \int_0^{0.5} \sqrt{1 + 4x^2}\,dx = \left(\frac{1}{2}x\sqrt{1 + 4x^2} + \frac{1}{4}\ln\left|2x + \sqrt{1 + 4x^2}\right|\right)\Big|_0^{0.5} = 0.57;\ L_2 = 0.91;\ L_3 = 1.35;\ L_4 = 1.82.$$

The parallels radii: $R_1 = 0.5$, $R_2 = 1$, $R_3 = 1.5$, $R_4 = 2$. According to the formula (2), the lengths of segment parallels arcs are equal to:

$$a_1 = \frac{2\pi R_1}{24} = \frac{2\pi \cdot 0.5}{24} = 0.13;\quad a_2 = 0.26;\quad a_3 = 0.39;\quad a_4 = 0.52.$$

On Fig. 2, a half of the segment surface is given with a 10-folds increase of all the sizes and we got full development of paraboloid.

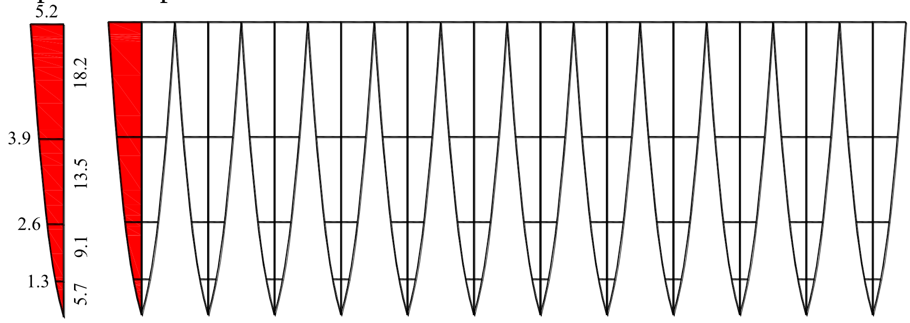

Fig. 2. Development of a rotational paraboloid

Building a development of a hyperboloid of one sheet. For example, we have a given hyperboloid of one sheet with the main meridian, which equation is $y=\frac{1}{x}$, and the axis of rotation is straight $y=-x$. The surface equation appears as: $\frac{x^2}{2}+\frac{y^2}{2}-\frac{z^2}{2}=1$.

Now we explore the curve $y=\frac{1}{x}$ on the xOy plane (Fig. 3). Since the curve $y=\frac{1}{x}$ is symmetric with respect to the line $y=x$, then we consider only the segment $x \in [0.4, 1]$, divided into several smaller ones: [0.4, 0.5], [0.5, 0.6], [0.6, 0.8], and [0.8, 1].

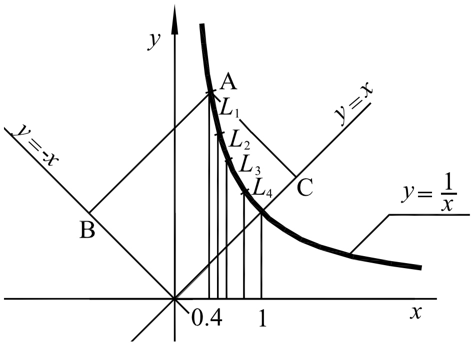

Fig. 3. Hyperboloid generatrix in the axes xOy

With the formula (1) we compile an integral for calculating the hyperbola arc length:

$$L=\int_a^b \sqrt{1+(f^{/}(x))^2}\,dx=\int_a^b \sqrt{1+((\frac{1}{x})^{/})^2}\,dx=\int_a^b \sqrt{1+\frac{1}{x^4}}\,dx=\int_a^b \frac{\sqrt{1+x^4}}{x^2}\,dx .$$

The integral as given above can not be resolved. But since we have the integration limits [0.4, 1], then we use the formula, allowing the integrated function $\frac{\sqrt{1+x^4}}{x^2}$ development into Maclauren series: $\frac{\sqrt{1+x^4}}{x^2}=\frac{1}{x^2}+\frac{1}{2}x^2-\frac{1}{8}x^6+\frac{1}{16}x^{10}-\frac{5}{128}x^{14}+\ldots$, at $x \in [-1, 1]$.

To integrate, we take only the first two products, as the following ones give very small values in the calculation and can be neglected. Finally, we have:

$$L=\int_a^b \frac{\sqrt{1+x^4}}{x^2}\,dx=\int_a^b \left(\frac{1}{x^2}+\frac{1}{2}x^2\right)dx=\left(-\frac{1}{x}+\frac{1}{2}\cdot\frac{x^3}{3}\right)\Big|_b^a .$$

Introducing the integration limits to the latest formula we get:

$$L_1 = \int_{0.4}^{0.5} \frac{\sqrt{1+x^4}}{x^2} = \left(-\frac{1}{x} + \frac{1}{2} \cdot \frac{x^3}{3} \right) \Bigg|_{0.4}^{0.5} = -\frac{1}{0.5} + \frac{0.5^3}{6} + \frac{1}{0.4} - \frac{0.4^3}{6} = 0.5;$$

$L_2 = 0.35$; $L_3 = 0.47$; $L_4 = 0.33$.

To estimate the parallel radius R_1 = AB, we have to define B, point of AB straight crossing to $y = -x$ (Fig. 2). Compiling the equation for AB:

$$y - 2.5 = x - 0.4 \quad \Rightarrow \quad y = x + 2.1.$$

We make a system:

$$\begin{cases} y = x + 2.1 \\ y = -x \end{cases} \quad \Rightarrow \quad B(-1.05, 1.05).$$

$$R_1 = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2} = \sqrt{(0.4 + 1.05)^2 + (2.5 - 1.05)^2} = 2.05.$$

Similarly, we find: $R_2 = 1.77$, $R_3 = 1.6$, $R_4 = 1.45$.

Lengths of the segment arcs:

$$a_1 = \frac{2\pi R_1}{24} = \frac{2\pi \cdot 2.05}{24} = 0.54; \quad a_2 = 0.46; \quad a_3 = 0.42; \quad a_4 = 0.38.$$

The surface neck radius; $R = \sqrt{2} = 1.41$, $a = \frac{2\pi R}{24} = \frac{2\pi \cdot 1.41}{24} = 0.37$.

On Fig. 4, a quarter of the surface segment was given with a 10-folds increase of all received values and a complete development of a hyperboloid.

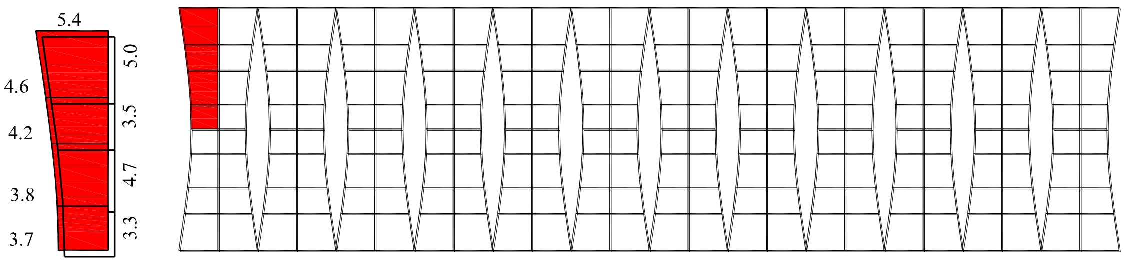

Fig. 4. Development of a hyperboloid of one sheet

Building a development of a hyperboloid of two sheets. The studied hyperbola $y = \frac{1}{x}$ when rotated around its real axis, will produce a two-cavity rotational hyperboloid according to the equation: $\frac{z^2}{2} - \frac{x^2}{2} - \frac{y^2}{2} = 1$.

Since the hyperbola arcs lengths are already found, we'll estimate the parallels radii and the arcs lengths. The largest parallel radius is R_1 = AC, and for its value determining it is necessary to find C, the AC straight to $y = x$ straight intersection point (Fig. 3). We formulate the equation of AC straight line: $y - 2.5 = -(x - 0.4) \quad \Rightarrow \quad y = x + 2.9$.

Making the system:

$$\begin{cases} y = x + 2.9 \\ y = x \end{cases} \quad \Rightarrow \quad C(1.45, 1.45).$$

$$R_1 = \sqrt{(x_A - x_C)^2 + (y_A - y_C)^2} = \sqrt{(0.4 - 1.45)^2 + (2.5 - 1.45)^2} = 1{,}49.$$

Similarly, we determine: $R_2 = 1.06$, $R_3 = 0.75$, $R_4 = 0.32$.

Lengths of the segment arcs: $a_1 = 0.39$; $a_2 = 0.28$; $a_3 = 0.2$; $a_4 = 0.08$.

On Fig. 5, half of the surface segment is given, with a 10-folds increase of all received sizes 10 times and a complete development of hyperboloid of two sheets.

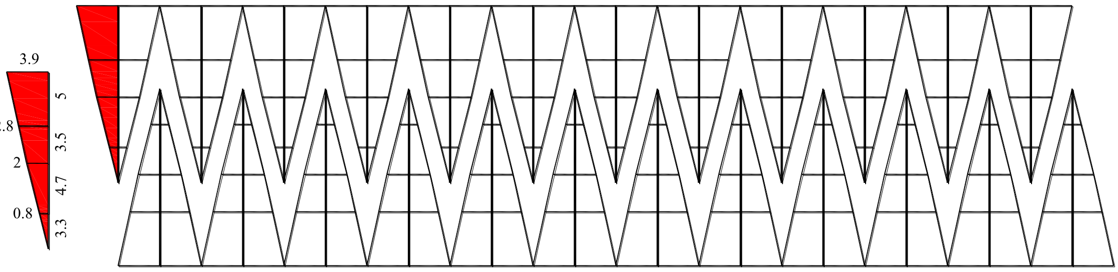

Fig. 5. Development of a hyperboloid of two sheets

Building a development of a catenoid. In the descriptive geometry course, the catenoid as a surface is not considered. As part of the learning process, it is represented as an arbitrary surface of rotation, but students do not build complex drawings of this surface. But in the advanced mathematics course attention is paid to this surface: students determine the generatrix length, volume, mean and Gaussian curvature, the first and second quadratic forms.

This surface is formed by rotating the chain $y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$ around the *Ox* axis. The catenoid belongs to the minimal surfaces (the principal curvatures sum is zero at all points). Such surface classic image is represented with the soap solution surface, tensioned on two wire circles.

To construct the development of catenoid, we use the main meridian curve equation: $y=\frac{e^x+e^{-x}}{2}$. The equation of the surface itself (Figure 6) has the following form:

$$\begin{cases} x=\dfrac{e^u+e^{-u}}{2}\cos v \\ y=\dfrac{e^u+e^{-x}}{2}\sin v \\ z=u \end{cases} \quad \text{or} \quad \begin{cases} x=chu\cos v \\ y=chu\sin v \\ z=u \end{cases}.$$

To integrate the curve, we take the segment $x \in [0, 1.5]$ and break it into segments: $x \in [1, 0.5], [0.5, 1.0], [1.0, 1.5]$ (Fig. 7).

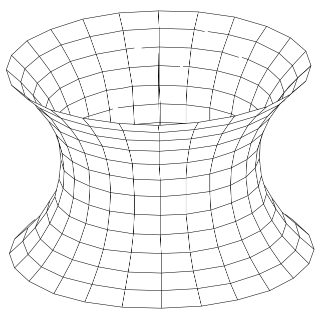

Fig. 6. Catenoid

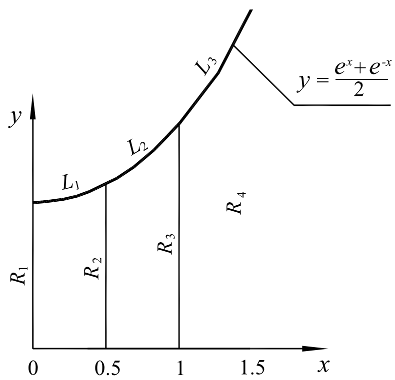

Fig. 7. Catenoid generatrix in the *xOy* axes

Now we construct an integral for calculating the chain line arc by the formula (1):

$$L=\int_a^b \sqrt{1+(f^{/}(x))^2}\,dx = \int_a^b \sqrt{1+\left(\left(\frac{e^x+e^{-e}}{2}\right)^{/}\right)^2}\,dx = \int_a^b \sqrt{1+\frac{e^{2x}}{4}-\frac{1}{2}+\frac{e^{-2x}}{4}}\,dx=$$

$$= \int_a^b \sqrt{\frac{1}{4}\left(e^x + e^{-x}\right)^2}\,dx = \frac{1}{2}\int_a^b (e^x + e^{-x})dx = \frac{1}{2}(e^x - e^{-x})\Big|_a^b .$$

And substitute the integration limits into the last formula:

$$L_1 = \frac{1}{2}\int_0^{0.5} (e^x + e^{-x})dx = \frac{1}{2}(e^x - e^{-x})\Big|_0^{0.5} = \frac{1}{2}\left(\sqrt{e} - \frac{1}{\sqrt{e}}\right) = 0.52;\quad L_2 = 0.65;\quad L_3 = 0.95.$$

The parallels radii represent the function values at marked points:

$$R_1 = y(0) = \frac{e^0 + e^0}{2} = 1.0;\quad R_2 = y(0.5) = 1.13;\quad R_3 = y(1.0) = 1.54;\quad R_4 = y(1.5) = 2.35.$$

With the formula (2) we find a_n: $a_1 = 0.26$; $a_2 = 0.3$; $a_3 = 0.4$; $a_4 = 0.61$.

On Fig. 8, a 10-folds increased quarter of the segment surface is given with a complete development of catenoid.

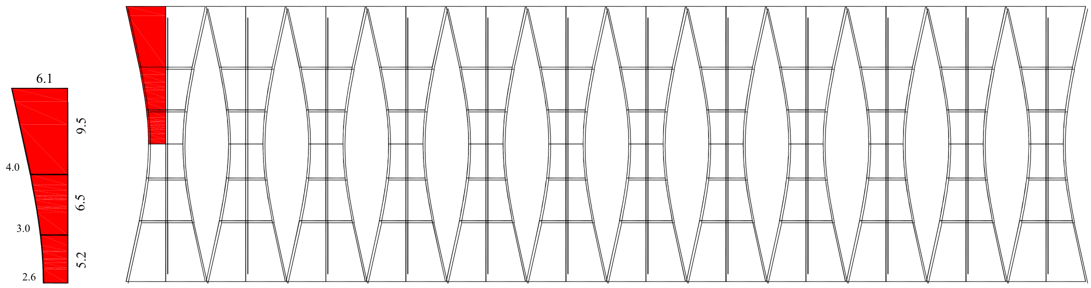

Fig. 8. Development of a cateniod

Construction of a pseudosphere development. The pseudosphere is a surface of constant negative curvature, which is formed by wrapping the tractrix around its asymptotes. This surface, as well as the catenoid surface, is considered only in the course of advanced mathematics.

Let we have a pseudosphere with a generatrix, whose equation for the *xOz* plane is (Fig. 9):

$$\begin{cases} x = \sin t \\ z = \ln tg\dfrac{t}{2} + \cos t \end{cases}.$$

The equation of the surface becomes:

$$\begin{cases} x = \sin t \cos v \\ y = \sin t \sin v \\ z = \ln tg\dfrac{t}{2} + \cos t \end{cases}.$$

Let's construct an integral for calculating the tractix arc length. Since the curve is given parametrically, we use the formula for calculating the curve arc length given parametrically (on the plane *xOz*) [2]:

$$L = \int_{t_1}^{t_2} \sqrt{(x')^2 + (z')^2} = \int_{t_1}^{t_2} \sqrt{\cos^2 t + \frac{\cos^4 t}{\sin^2 t}} = \int_{t_1}^{t_2} ctgt = \ln\left|\sin t\right|_{t_1}^{t_2} .$$

In our case, the parameter t is found in the interval $[\,t_1; \frac{\pi}{2}\,]$. We divide this interval into several ones : $t \in [\frac{\pi}{30}; \frac{\pi}{20}],[\frac{\pi}{20}; \frac{\pi}{12}],[\frac{\pi}{12}; \frac{\pi}{8}], [\frac{\pi}{8}; \frac{\pi}{6}],[\frac{\pi}{6}; \frac{\pi}{4}],[\frac{\pi}{4}; \frac{\pi}{2}]$ (Fig. 10).

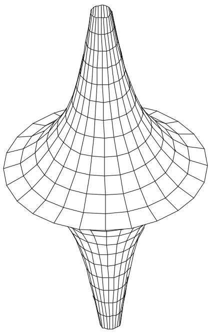

Fig. 9. Pseudosphere

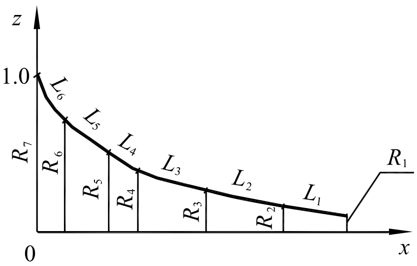

Fig. 10. Pseudosphere generatrix in the *xOy* axes

Now we find the lengths of arcs at these intervals:

$$L_1 = \int_{\frac{\pi}{30}}^{\frac{\pi}{20}} ctgt = \ln\left|\sin t\right|\Big|_{\frac{\pi}{30}}^{\frac{\pi}{20}} = \ln\left|\sin\frac{\pi}{20}\right| - \ln\left|\sin\frac{\pi}{30}\right| = -1.855 - (2.258) = 0.403;$$

$$L_2 = 0.503;\ L_3 = 0.39,\ L_4 = 0.27,\ L_5 = 0.35,\ L_6 = 0.35.$$

The parallels radii are equals to the value $z(t)$:

$$R_1 = z\left(\frac{\pi}{30}\right) = \ln tg\frac{\pi}{60} + \cos\frac{\pi}{30} = 0.1, \quad a_1 = 0.026;$$

$R_2 = 0.16, \quad a_2 = 0.042;$ $\qquad R_3 = 0.26,\ a_3 = 0.068;$

$R_4 = 0.38, \quad a_4 = 0.1;$ $\qquad R_5 = 0.5, \quad a_5 = 0.13;$

$R_6 = 0.71, \quad a_6 = 0.18;$ $\qquad R_7 = 1.0, \quad a_7 = 0.26.$

On Fig. 11 a quarter of the segment surface is represented with 10-folds increase of all obtained values and a complete development of a pseudosphere.

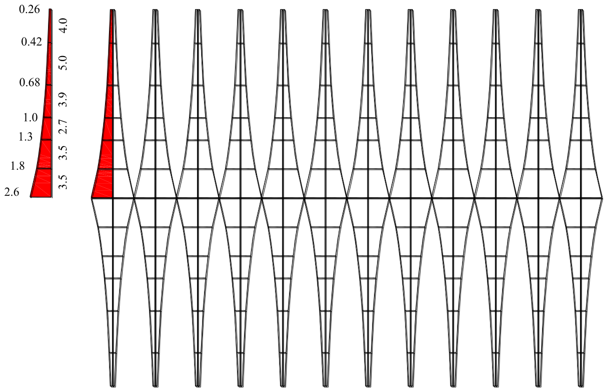

Fig. 11. Development of a pseudosphere

Conclusion. The surfaces developments are widely used for architectural modeling of projects. The Architectural Institute students of Odessa State Academy of Building and Architecture (Ukraine) do study such discipline as «Volumetric-Spatial Compositions» at the Department of «Architectural Design Fundamentals» where during two terms they train to make paper models of their projects. The students use relatively simple geometric elements: planes, prisms, cylinders, etc. to express their architectural decisions in paper and card models (Fig. 12).

But today, modern architects are trying to diversify the architectural environment with more complex surfaces [5]. For example, in Fig. 13 there are photos of modern buildings, which have the form of curvilinear surfaces.

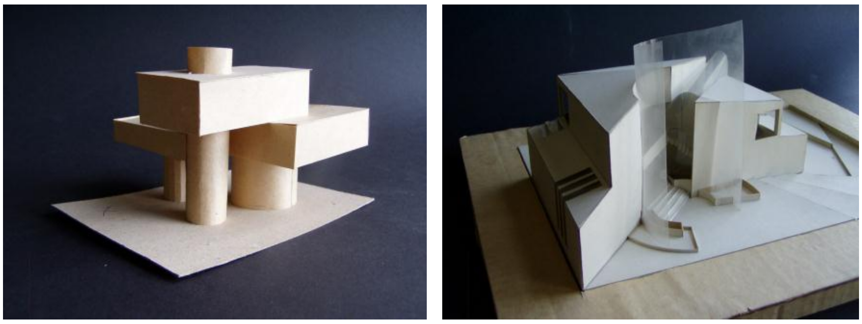

Fig. 12. Models of students

Fig. 13. Modern buildings (Soumaya Museum, Agbar tower, Khan Shatyr)

Accordingly, students also try to complicate their paper models with curvilinear forms. In our opinion the proposed rotational surface involutes can be used in this capacity (Fig. 14). Certainly, it's not just simple copying the design of building, it's about having some elements be efficiently used.

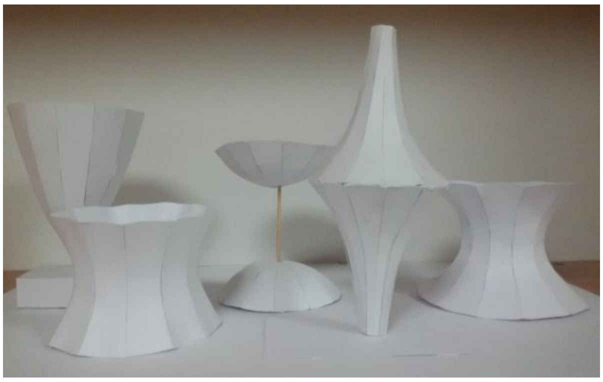

Fig. 14. The paper models of surfaces considered in the article

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Стаття надійшла 26.03.2018