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PERTURBED MOTIONS OF A RIGID BODY, CLOSE TO THE LAGRANGE CASE

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The perturbed motions of a rigid body, close to the Lagrange case, are investigated. Conditions are presented for the possibility of averaging the equations of motion with respect to the nutation angle and the averaged system of equations is obtained. An actual mechanical model of the perturbations, corresponding to the body's motion in a medium with linear dissipation, is considered. A numerical solution of the averaged system is constructed.

1. Original equations and the unperturbed motion. Perturbed motion relative to a fixed point of a dynamically symmetric heavy rigid body can be analyzed in the case of perturbations of arbitrary nature. The equations of motion are

$$
\begin{aligned}
&Ap^{\cdot} + (C-A)qr = mgl\sin\theta\cos\varphi + \varepsilon M_1 \qquad (1.1)\\
&Aq^{\cdot} + (A-C)pr = -mgl\sin\theta\sin\varphi + \varepsilon M_2\\
&Cr^{\cdot} = \varepsilon M_3, \quad M_i = M_i(p, q, r, \varphi, \theta, \psi), \quad i = 1,2,3\\
&\varphi^{\cdot} = r - (p\sin\varphi + q\cos\varphi)\operatorname{ctg}\theta\\
&\theta^{\cdot} = p\cos\varphi - q\sin\varphi, \quad \psi^{\cdot} = (p\sin\varphi + q\cos\varphi)\operatorname{cosec}\theta
\end{aligned}
$$

The dynamic Eqs. (1.1) have been written in projections onto the body's principle axes of inertia. Here p, q and r are the projections of the angular velocity vector onto these axes; εM_i $(i = 1, 2, 3)$ are the projections of the vector of perturbing moments onto the same axes; φ, θ and ψ are the Euler angles [1]; ε is a small parameter characterizing the magnitude of the perturbations (in particular, when $\varepsilon = 0$ the system (1.1) describes motion in the Lagrange case [1]); m is the body's mass; g is the gravitational acceleration; l is the distance from the fixed point O to the body's center of gravity; A and C are the body's respectively, equatorial and axial moments of inertia relative to the fixed point O.

The problem is posed of investigating the behavior of the solution of system (1.1) for nonzero values of the small parameter ε on a sufficiently long time interval $t \sim \varepsilon^{-1}$. The averaging method [2,3] is used for solving the problem.

Averaging with respect to the Euler — Poinsot motion of an asymmetric rigid body was first carried out in [4]. In contrast to the procedure of averaging with respect to the Euler — Poinsot motion, averaging with respect to the Lagrange motion permits us to examine the motion with external force moments, large in absolute value, as the generating solution. Papers [5 — 8] were devoted to the investigation of perturbed motions close to Lagrange motion. The method of reference motions was used in [5].

In [6] the averaging method was applied for a special kind of generating solution. A numerical averaging procedure was suggested in [7]. The motion of a body differing slightly from a Lagrange gyroscope was studied in [8] with the aid of the theorem on the preservation of motions.

Below we develop a averaging procedure for system (1.1) under arbitrary initial conditions for perturbations admitting of averaging with respect to the nutation angle θ. The error in the averaged solution for the slow variables is of the order of ε on the time interval in which the body accomplishes $\sim\varepsilon^{-1}$ revolutions. At first we derive the necessary relations for the unperturbed motion [1], i.e., when $\varepsilon = 0$. The first integrals of the equations of motion for the unperturbed system (1.1) are

$$G_z = A \sin\theta\,(p\sin\varphi + q\cos\varphi) + Cr\cos\theta = c_1 \tag{1.2}$$
$$H = {}^1/_2\,[A\,(p^2+q^2) + Cr^2] + mgl\cos\theta = c_2, \quad r = c_3$$

where G_z is the projection of the kinetic moment vector onto the vertical Oz, H is the body's total energy, r is the projection of the angular velocity vector onto the axis of dynamic symmetry, $c_i\ (i = 1, 2, 3)$ are arbitrary constants $(c_2 \geqslant -mgl)$.

The expression for the nutation angle θ in the unperturbed motion as a function of time t, of the motion integrals (1.2), and of an arbitrary phase constant β is well known [1]

$$u = \cos\theta = u_1 + (u_2 - u_1)\,\mathrm{sn}^2(\alpha t + \beta) \tag{1.3}$$
$$\alpha = [mgl\,(u_3 - u_1)\,/\,(2A)]^{1/2}$$
$$\mathrm{sn}\,(\alpha t + \beta) = \sin\,\mathrm{am}\,(\alpha t + \beta,\ k)$$
$$k^2 = (u_2 - u_1)(u_3 - u_1)^{-1}, \quad 0 \leqslant k^2 \leqslant 1 \tag{1.4}$$

where sn is the elliptic sine [9], k is the modulus of ellipticity of the functions, and u_1, u_2 and u_3 are real roots of the cubic polynomial

$$Q\,(u) = A^{-2}\,[(2H - Cr^2 - 2mglu)(1 - u^2)A - (G_z - Cru)^2], \quad -1 \leqslant u_1 \leqslant u_2 \leqslant 1 \leqslant u_3 < +\infty \tag{1.5}$$

Relations between the roots of the polynomial $Q\,(u)$ of (1.5) and first integrals (1.2) can be written, according to Vieta's theorem, in the following manner

$$u_1 + u_2 + u_3 = \frac{H}{mgl} - \frac{Cr^2}{2mgl} + \frac{C^2r^2}{2Amgl} \tag{1.6}$$
$$u_1u_2 + u_1u_3 + u_2u_3 = \frac{G_zCr}{Amgl} - 1$$
$$u_1u_2u_3 = -\frac{H}{mgl} + \frac{Cr^2}{2mgl} + \frac{G_z^2}{2Amgl}$$

The variables φ and ψ are obtained by quadratures from the following equations [1]

$$\varphi^{\cdot} = r - \frac{(G_z - Cru)\,u}{A\,(1-u^2)}, \quad \psi^{\cdot} = \frac{G_z - Cru}{A\,(1-u^2)} \tag{1.7}$$

Function u specified by relation (1.3) depends upon four constants of integration G_z, H, r and β. The subsequent integration of Eqs. (1.7) yields two more arbitrary constants. Thus, formulas (1.2) — (1.7) describe the general solution of system (1.1) when $\varepsilon = 0$.

2. The averaging procedure. Let us reduce the equations of perturbed motion (1.1) to a form admitting of the application of the averaging method [2,3]. To do this we pick out the slow and the fast variables. In the problem being analyzed the first integrals (1.2) are the slow variables for perturbed motion (1.1). The fast variables are the angles of natural gyration φ, of nutation θ, and of precession ψ. Using relations (1.2) as the formulas for transforming the variables $(p, q, r, \varphi, \theta, \psi)$ to the variables $(G_z, H, r, \varphi, \theta, \psi)$, we reduce the first three equations in (1.1) to

$$G_z^{\cdot} = \varepsilon[(M_1 \sin \varphi + M_2 \cos \varphi) \sin \theta + M_3 \cos \theta] \qquad (2.1)$$
$$H^{\cdot} = \varepsilon (M_1 p + M_2 q + M_3 r)$$
$$r^{\cdot} = \varepsilon C^{-1} M_3, \quad M_i = M_i (p, q, r, \varphi, \theta, \psi), \quad i = 1, 2, 3$$

Here and in the last three (kinematic) equations in (1.1) it is implicit that the variables p, q, r have been expressed as functions of $G_z, H, r, \varphi, \theta, \psi$ by using (1.2) and have been substituted into (1.1) and (2.1). The initial values of the slow variables G_z, H, r can be computed from (1.2).

The right hand sides of Eqs. (2.1) contain the three fast variables, which presents a difficulty for the application of the averaging method, connected with the possibility of the appearance of nonlinear resonances. To eliminate this difficulty we require that the right hand sides of the Eqs. (2.1) for the slow variables depend, in fact, on only one fast variable, viz., the nutation angle θ and be periodic functions of θ of period 2π. Then Eqs. (2.1) can be averaged with respect to θ and the equations of first approximation obtained. It happens that a number of applied problems possess the property indicated and admit of averaging with respect to one variable, viz., the nutation angle θ.

Let us derive certain sufficient conditions for the possibility of averaging Eqs. (2.1) only with respect to the nutation angle θ. Under fixed values of the slow variables the right hand sides of Eqs. (2.1) being averaged contain combinations of the following kind

$$M_1 \sin \varphi + M_2 \cos \varphi, \quad M_1 p + M_2 q, \quad M_3$$

We require that these combinations be represented, using identity transformations, as functions of the slow variables and of the nutation angle θ, 2π-periodic in θ, i.e.,

$$M_1 \sin \varphi + M_2 \cos \varphi = M_1^* (G_z, H, r, \theta) \qquad (2.2)$$
$$M_1 p + M_2 q = M_2^* (G_z, H, r, \theta)$$
$$M_3 = M_3^* (G_z, H, r, \theta)$$

Note that the equalities

$$p \sin \varphi + q \cos \varphi = (G_z - Cr \cos \theta) / (A \sin \theta) \qquad (2.3)$$
$$p^2 + q^2 = [2(H - mgl \cos \theta) - Cr^2] / A$$

ensure from relations (1.2). Consequently, combinations of form (2.3) are expressed only in terms of the slow variables and the nutation angle θ, and they are 2π-periodic in θ. Therefore, the combinations (2.3) are reduced to form (2.2). Setting up relations (2.2) and (2.3), we see that if the perturbing moments M_i satisfy the conditions

$$M_1 = pf, \quad M_2 = qf, \quad M_3 = M_3^* \qquad (2.4)$$

or the conditions

$$M_1 = F \sin \varphi, \quad M_2 = F \cos \varphi, \quad M_3 = M_3^* \tag{2.5}$$

where the arbitrary functions f, F and M_3^* have the form

$$\begin{aligned} &f = f(G_z, H, r, \theta), \quad F = F(G_z, H, r, \theta) \\ &M_3^* = M_3^*(G_z, H, r, \theta) \end{aligned} \tag{2.6}$$

and are 2π -periodic in θ, then the requirements (2.2) imposed are fulfilled. Thus, for the perturbed Lagrange motion the requirements (2.4) or (2.5) imposed on the moments of the forces applied are sufficient conditions for the possibility of averaging the Eqs. (2.1) for the slow variables with respect to the nutation angle θ. In what follows we assume the fulfilment of the general (necessary and sufficient) conditions (2.2) or, in particular, of the sufficient conditions (2.4) or (2.5) (together with (2.6)), which ensures the validity of relations (2.2). Then system (2.1) can be presented in the form

$$\begin{aligned} G_z^{\cdot} &= \varepsilon F_1(G_z, H, r, \theta), \quad F_1 = M_1^* + M_3^* \cos \theta \\ H^{\cdot} &= \varepsilon F_2(G_z, H, r, \theta), \quad F_2 = M_2^* + M_3^* r \\ r^{\cdot} &= \varepsilon F_3(G_z, H, r, \theta), \quad F_3 = C^{-1} M_3^* \end{aligned} \tag{2.7}$$

Here F_1, F_2, F_3 are 2π -periodic functions of θ.

We propose to carry out the investigation of the perturbed motion in the slow variables u_i $(i = 1, 2, 3)$, connected with G_z, H and r by relations (1.6) and more convenient for analysis. The cubic Eq. (1.5) need not be solved relative to u_i. The slow variables G_z, H and r can be expressed in terms of u_i from (1.6) as follows:

$$\begin{aligned} G_z &= (Amgl)^{1/2}(u_1 + u_2 + u_3 + u_1u_2u_3 + \delta_1 R)^{1/2} \times \\ &\quad \delta_2 \operatorname{sign}(1 + u_1u_2 + u_1u_3 + u_2u_3) \\ H &= {}^1/_2\, mgl\, [(u_1 + u_2 + u_3)(1 + A/C) + (\delta_1 R - u_1u_2u_3) \times \\ &\quad (1 - A/C)] \\ r &= \delta_2 C^{-1}(Amgl)^{1/2}(u_1 + u_2 + u_3 + u_1u_2u_3 - \delta_1 R)^{1/2} \\ R &= [(1 - u_1^2)(1 - u_2^2)(u_3^2 - 1)]^{1/2} \\ \delta_1 &= \operatorname{sign}(G_z^2 - C^2 r^2), \quad \delta_2 = \operatorname{sign} r \end{aligned} \tag{2.8}$$

At the initial instant the quantities δ_1 and δ_2 are determined from the initial conditions for G_z and r. If during the motion one or both of the quantitites $G_z^2 - C^2r^2$ and r pass through zero, a change of sign is possible for δ_1 and δ_2, to determine which we can make use of the original system (2.7). We write relations (1.6) in the abbreviated form

$$S_i(u_1, u_2, u_3) = \Phi_i(G_z, H, r), \quad i = 1, 2, 3 \tag{2.9}$$

where S_i and Φ_i are known functions of their arguments (see (1.6)). Differentiating (2.9) with respect to time and substituting expressions (2.7) in the place of $G_z^{\cdot}$, $H^{\cdot}$, $r^{\cdot}$, we obtain the relations

$$\sum_{j=1}^{3} \frac{\partial S_i}{\partial u_j} u_j^{\cdot} = \varepsilon Z_i(u_1, u_2, u_3, \theta), \quad i = 1, 2, 3 \tag{2.10}$$

Here

$$Z_i = \frac{\partial \Phi_i}{\partial G_z} F_1 + \frac{\partial \Phi_i}{\partial H} F_2 + \frac{\partial \Phi_i}{\partial r} F_3, \quad i = 1, 2, 3 \tag{2.11}$$

Further, we need to solve the linear Eqs. (2.10) relative to the derivatives $u_i^{\cdot}$. The determinant D of the linear system (2.10) equals

$$D = \det(\partial S_i / \partial u_j) = (u_1 - u_2)(u_1 - u_3)(u_2 - u_3)$$

and is nonzero by assumption. The partial derivatives in (2.11) can be represented, by using equalities (2.8), as functions of only the variables u_i. Thus, after solving system (2.10) relative to the derivatives, the desired system of equations for the slow variables u_i takes the form, analogous to (2.7),

$$\begin{aligned} &u_i^{\cdot} = \varepsilon V_i(u_1, u_2, u_3, \theta), \quad i = 1, 2, 3 \\ &V_i = V_{i1}F_1^* + V_{i2}F_2^* + V_{i3}F_3^*, \qquad V_{ij} = V_{ij}(u_1, u_2, u_3), \\ &j = 1, 2, 3 \end{aligned} \tag{2.12}$$

where we have

$$\begin{aligned} V_{11} &= \frac{G_z - Cru_1}{Amgl(u_1 - u_2)(u_1 - u_3)} \\ V_{12} &= \frac{u_1^2 - 1}{mgl(u_1 - u_2)(u_1 - u_3)} \\ V_{13} &= \frac{C}{mgl(u_1 - u_2)(u_1 - u_3)} \left[\left(\frac{C}{A} - 1 \right) ru_1^2 - \frac{G_z}{A} u_1 + r \right] \end{aligned} \tag{2.13}$$

Here, instead of G_z and r we must substitute the corresponding formulas (2.8). The functions V_{2j} and V_{3j} ($j = 1, 2, 3$) are obtained from the corresponding expressions in (2.13) for the same values of j by cyclic permutation of the indices on the quantities u_i. The functions F_i^* are obtained by substituting into the F_i from (2.7) the expressions (2.8) for the integrals. The initial values u_i° for variables u_i are computed from the initial data G_z°, H°, r° with the aid of relations (1.6).

The procedure for averaging the first-approximation Eqs. (2.12) for the slow variables u_i is the following [2,3]. Into the right hand side of system (2.12) we substitute the fast variable θ from expression (1.3) for the unperturbed motion

$$\theta = \arccos[u_1 + (u_2 - u_1)\,\mathrm{sn}^2(\alpha t + \beta)] \tag{2.14}$$

After the substitution of (2.14) the right hand sides of system (2.12) are periodic functions of t with period $2\mathbf{K}(k) / \alpha$, where k and α are defined by relations (1.3) and (1.4). Averaging the right hand sides of the resultant system with respect to t, we obtain, in the slow time $\tau = \varepsilon t$, the averaged system of first approximation (differentiation with respect to τ is denoted by a prime, while the previous notation is retained for the averaged variables)

$$u_i' = U_i(u_1, u_2, u_3), \quad u_i(0) = u_i^\circ, \quad i = 1, 2, 3 \tag{2.15}$$

Here

$$U_i(u_1, u_2, u_3) = \frac{\alpha}{2\mathbf{K}(k)} \int_0^{2\mathbf{K}/\alpha} V_i(u_1, u_2, u_3, \theta(t))\, dt \tag{2.16}$$

and expression (2.14) has been substituted as $\theta = \theta(t)$ in (2.16).

Thus, according to the procedure suggested, the investigation of perturbed Lagrange motion is carried out in the following way. Let the perturbing moments εM_i satisfy conditions (2.2) or, in particular, (2.4) or (2.5) (together with (2.6)). We compute in succession the functions M_i^*, F_i^*, V_i $(i = 1, 2, 3)$, using relations (2.2), (2.7), (2.12), (2.13). After this we average V_i in accord with (2.16), using expressions (1.3), (1.4), (2.14), and we set up the averaged system (2.15). System (2.15) is significantly simpler than the original system (1.1) since it is of lesser order (third instead of sixth), is autonomous, and does not contain fast oscillations. After investigating and solving system (2.15) for u_i, the original slow variables G_z, H, r are recovered from formulas (2.8). The fast variables φ, θ, ψ can be found by using relations (1.7) and (2.14). In accord with the general theorems of the averaging method [2,3] the slow variables u_i or G_z, H, r are determined with an error of order ε, while the fast variables are determined with an error of order unity on an interval of order ε^{-1} of variation of time t.

3. Perturbed motion of a rigid body under linear dissipative moments. As an example of the procedure worked out we investigate the perturbed Lagrange motion with due regard to the moments acting on a rigid body from the surrounding medium. We take the perturbing moments εM_i $(i = 1, 2, 3)$ in the form [10]

$$M_1 = -ap, \ M_2 = -aq, \ M_3 = -br, \ a, \ b > 0 \tag{3.1}$$

where a and b are certain constant proportionality coefficients depending on the medium's properties and the body's shape. Moments (3.1) satisfy the sufficient conditions (2.4) and (2.6) for the possibility of averaging with respect to only the nutation angle θ. System (2.1) can be written as follows

$$\begin{aligned} G_z^{\cdot} &= -\varepsilon\,[a\,(p\sin\varphi + q\cos\varphi)\sin\theta + br\cos\theta] \\ H^{\cdot} &= -\varepsilon\,[a\,(p^2+q^2) + br^2] \\ r^{\cdot} &= -\varepsilon\, bC^{-1} r \end{aligned} \tag{3.2}$$

Having integrated the third equation in (3.2), we obtain (r° is an arbitrary initial value of the axial rotation velocity)

$$r = r^\circ \exp(-\varepsilon b C^{-1} t) \tag{3.3}$$

In accord with the procedure in Sect. 2 we pass to the new slow variables u_i and we average in accord with (2.16). We obtain an averaged system (2.15) in slow time $\tau = \varepsilon t$ of form

$$\begin{aligned} u_1' = &-\frac{1}{mgl\,(u_1-u_2)(u_1-u_3)}\Big[b\frac{C}{A}r^2u_1^2 - a\frac{C}{A}r^2(u_1^2-1) - \\ &2\frac{a}{A}mgl\,(u_1^2-1)\,v + 2\frac{a}{A}H\,(u_1^2-1) - \frac{C}{A}\Big(b - a\frac{C}{A}\Big)r^2u_1v - \\ &\frac{C}{A}\Big(\frac{a}{A}+\frac{b}{C}\Big)G_zru_1 + \frac{a}{A^2}G_z^2 + \frac{1}{A}\Big(b - a\frac{C}{A}\Big)G_zrv\Big] \\ v = &\; u_3 - (u_3-u_1)\,\mathbf{E}(k)\,/\,\mathbf{K}(k) \end{aligned} \tag{3.4}$$

Here the expressions (1.4) and (2.8) are substituted in the place of G_z, H, r, k. The equations for u_2 and u_3 are obtained from (3.4) by a cyclic permutation of the indices on u_i. However, under permutation the expression for v, wherein $\mathrm{K}(k)$ and $\mathrm{E}(k)$ are the complete elliptic integrals of the first and second kinds, should remain unchanged in all three equations.

The averaged system (3.4) was integrated numerically on a computer for $\tau \geqslant 0$ under various initial conditions and problem parameters. Let us present the calculation results for three cases corresponding to the following initial data:

	u_1°	u_2°	u_3°	θ°
1)	0.913	0.996	1.087	5°
2)	0	0.5	2	60°
3)	−0.992	−0.985	2.992	170°

The data presented correspond to a spinning top receiving at the initial instant an angular rotation velocity equal to $r^\circ = \sqrt{3}$ around the dynamic symmetry axis and deviated from the vertical by the angle θ°. In addition, we take $A = 1.5, C = 1$, $a = 0.125$, $b = 0.1$, $mgl = 0.5$. Using the values of u_i found as a result of the numerical integration, we determine the variables G_z, H, r from formulas (2.8).

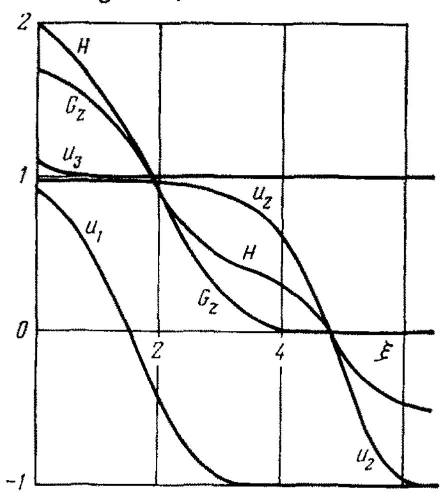

Fig. 1

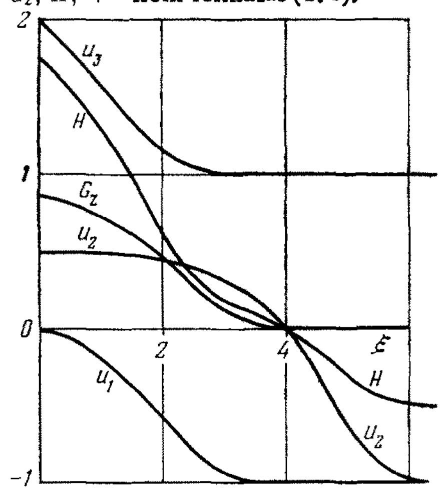

Fig. 2

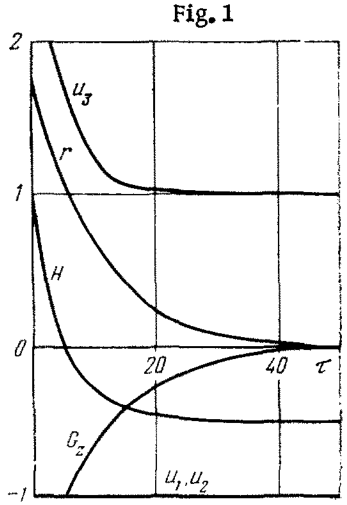

Fig. 3

The graphs of functions u_i $(i = 1, 2, 3)$, G_z, H, r are shown in Figs. 1 – 3 for the three cases mentioned. The total energy H decreases monotonically and asymptotically approaches the value $H = -mgl = -0.5$. The projection of the kinetic moment vector onto the vertical G_z in cases *1* and *2* decreases monotonically, while in case *3* it increases monotonically, tending to zero in all three cases. The quantities u_1 and u_2 decrease monotonically and tend to -1, while u_3 asymptotically approaches $+1$. In this connection, as follows from (1.3), we have $\cos\theta \to -1$ $(\theta \to \pi)$.

Thus, under the action of external dissipation the rigid body, for any initial condition, tends to the unique stable (lower) equilibrium position. In cases *1* and *2* the decrease of the variables takes place very slowly. Therefore, in these cases it becomes convenient to make the change of independent slow variable $\xi = \ln(1+\tau)$ in system (3.4). The correctness of the calculation was monitored by the fact that the values of r as obtained from the numerical data and from formulas (2.8) practically coincided with the exact solution (3.3).

An averaging procedure can be carried out analogously for the motion of a rigid body in the Lagrange case with a cavity filled by a liquid of high viscosity. The corresponding formulas for the moments of the viscous forces were obtained in [11].

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