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RAPID ROTATION OF A HEAVY GYROSTAT ABOUT A FIXED POINT IN A RESISTING MEDIUM

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UDC 531.1

We consider an asymmetric heavy rigid body with a spherical cavity filled with a high-viscosity liquid rotating rapidly about a fixed point in a weakly resisting medium. We call motions rapid when the moment of the applied forces about the fixed point is small in comparison with the instantaneous value of the kinetic energy of rotation.

To investigate the motion of the body with a liquid we introduce three Cartesian coordinate systems: a stationary x_i system ($i = 1, 2, 3$); a y_i system with the y_3 axis along the angular momentum vector **G** of the body and liquid (Fig. 1); a z_i system whose axes coincide with the principal axes of inertia of the rigid body. The x_i system is transformed into the y_i system by two rotations: by an angle λ about x_3, and by an angle δ about y_2. The position of the z_i axes with respect to the y_i system is determined by the Eulerian angles θ, φ, and ψ.

Table 1 lists the cosines of the angles between axes.

The equations of motion of an asymmetric body with respect to a fixed point have the following form [13]:

$$\frac{dG}{dt} = L_3;\quad \frac{d\delta}{dt} = \frac{L_1}{G};\quad \frac{d\lambda}{dt} = \frac{L_2}{G\sin\delta};$$
$$\frac{d\theta}{dt} = G\sin\theta\sin\varphi\cos\varphi\left(\frac{1}{A}-\frac{1}{B}\right)+\frac{L_2\cos\psi - L_1\sin\psi}{G};$$
$$\frac{d\varphi}{dt} = G\cos\theta\left(\frac{1}{C}-\frac{\sin^2\varphi}{A}-\frac{\cos^2\varphi}{B}\right)+\frac{L_1\cos\psi + L_2\sin\psi}{G\sin\theta};\qquad (1)$$
$$\frac{d\psi}{dt} = G\left(\frac{\sin^2\varphi}{A}+\frac{\cos^2\varphi}{B}\right)-\frac{L_1\cos\psi + L_2\sin\psi}{G}\operatorname{ctg}\theta - \frac{L_2}{G}\operatorname{ctg}\delta.$$

Here the L_i are the y_i components of the moment of the applied forces; G is the magnitude of the angular momentum; A, B, and C are the principal moments of inertia with respect to the z_i axes; θ, φ, and ψ are the Eulerian angles.

TABLE 1

Axis	Oz_1	Oz_2	Oz_3
Oy_1	$\alpha_{11} = \cos\varphi\cos\psi - \cos\theta\sin\varphi\sin\psi$	$\alpha_{12} = -\sin\varphi\cos\psi - \cos\theta\cos\varphi\sin\psi$	$\alpha_{13} = \sin\theta\sin\psi$
Oy_2	$\alpha_{21} = \cos\varphi\sin\psi + \cos\theta\sin\varphi\cos\psi$	$\alpha_{22} = -\sin\varphi\sin\psi + \cos\theta\cos\varphi\cos\psi$	$\alpha_{23} = -\sin\theta\cos\psi$
Oy_3	$\alpha_{31} = \sin\theta\sin\varphi$	$\alpha_{32} = \sin\theta\cos\varphi$	$\alpha_{33} = \cos\theta$

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Translated from Prikladnaya Mekhanika, Vol. 18, No. 7, pp. 102-107, July, 1982. Original article submitted January 14, 1980.

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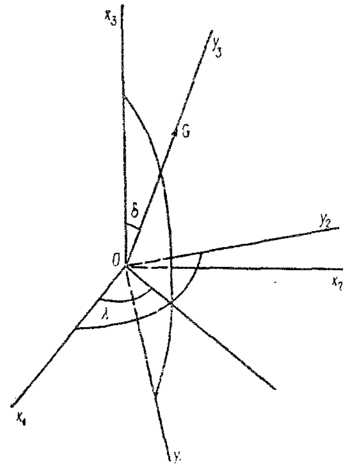

Fig. 1

Using the expressions for the z_i components of the angular momentum vector G

$$Ap = G\sin\theta\sin\varphi; \quad Bq = G\sin\theta\cos\varphi; \quad Cr = G\cos\theta, \tag{2}$$

we obtain for the kinetic energy T of the body and its time derivative

$$T = \frac{G^2}{2}\left[\left(\frac{\sin^2\varphi}{A} + \frac{\cos^2\varphi}{B}\right)\sin^2\theta + \frac{\cos^2\theta}{C}\right]; \tag{3}$$

$$T^{\cdot} = \frac{2T}{G}L_3 + G\sin\theta\left[\cos\theta\left(\frac{\sin^2\varphi}{A} + \frac{\cos^2\varphi}{B} - \frac{1}{C}\right)(L_2\cos\psi - L_1\sin\psi)\right.$$
$$\left. + \sin\varphi\cos\varphi\left(\frac{1}{A} - \frac{1}{B}\right)(L_1\cos\psi + L_2\sin\psi)\right], \tag{4}$$

where p, q, and r are the z_i components of the absolute angular velocity vector ω of the body.

By taking account of (2), the y_i components of the moment of the gravitational forces, the external resistance, and the components of the perturbing moment due to the effect of the viscous liquid in the cavity on the motion of the rigid body can be written in the form

$$L_1 = -mg\cos\delta\sum_{i=1}^{3} a_i\alpha_{2i} - G\sum_{i=1}^{3}\left(\frac{I_{1i}}{A}\alpha_{31}\alpha_{11} + \frac{I_{2i}}{B}\alpha_{32}\alpha_{12} + \frac{I_{3i}}{C}\alpha_{33}\alpha_{13}\right)$$
$$+ \frac{\rho PG^3}{\nu ABC}\left[\frac{(A-C)(A+C-B)}{AC}(\cos\psi\sin\theta\cos^2\theta\sin\varphi\cos\varphi\right.$$
$$-\sin\psi\sin\theta\cos\theta\sin^2\varphi) + \frac{(A-B)(A+B-C)}{AB}\cos\psi\sin^3\theta\sin\varphi\cos\varphi$$
$$\left.-\frac{(B-C)(B+C-A)}{BC}(\cos\psi\sin\theta\cos^2\theta\sin\varphi\cos\varphi + \sin\psi\sin\theta\cos\theta\cos^2\varphi)\right];$$
$$L_2 = mg\sum_{i=1}^{3} a_i(\alpha_{3i}\sin\delta + \alpha_{1i}\cos\delta) - G\sum_{i=1}^{3}\left(\frac{I_{1i}}{A}\alpha_{31}\alpha_{21} + \frac{I_{2i}}{B}\alpha_{32}\alpha_{22}\right.$$
$$\left.+ \frac{I_{3i}}{C}\alpha_{33}\alpha_{23}\right) + \frac{\rho PG^3}{\nu ABC}\left[\frac{(A-C)(A+C-B)}{AC}(\cos\psi\sin\theta\cos\theta\sin^2\varphi\right. \tag{5}$$
$$+\sin\psi\sin\theta\cos^2\theta\sin\varphi\cos\varphi) + \frac{(A-B)(A+B-C)}{AB}\sin\psi\sin^3\theta\sin\varphi\cos\varphi$$
$$\left.+\frac{(B-C)(B+C-A)}{BC}(\cos\psi\sin\theta\cos\theta\cos^2\varphi - \sin\psi\sin\theta\cos^2\theta\sin\varphi\cos\varphi)\right];$$
$$L_3 = -mg\sin\delta\sum_{i=1}^{3} a_i\alpha_{2i} - G\sum_{i=1}^{3}\left(\frac{I_{1i}}{A}\alpha_{31}^2 + \frac{I_{2i}}{B}\alpha_{32}^2 + \frac{I_{3i}}{C}\alpha_{33}^2\right).$$

Here the I_{ij} are the aerodynamic dissipative torque coefficients [1], which are assumed constant.

Since we are studying rapid motion, the ratio $mga/T_0 \sim \varepsilon \ll 1$ is assumed small, where a is the distance from the center of mass to the fixed point. The resistance of the medium is assumed weak and of the same order of smallness: $\|I\|/G_0 \sim \varepsilon \ll 1$, where $\|I\|$ is the norm of the dissipative coefficients matrix.

Terms taking account of the effect of the viscous liquid filling the cavity on the motion of the rigid body are derived in [14] by using the table of direction cosines, where ρ is the density of the liquid, and ν is the kinematic viscosity. The constant tensor P depends only on the shape of the cavity, and characterizes the dissipation of energy resulting from the viscosity of the liquid.

In the problem under consideration we specify the tensor P in the form $P_{ij} = P\delta_{ij}$, where δ_{ij} is the Kronecker symbol, and $P > 0$. Thus, for example, for a spherical cavity of radius a we have $P = 8\pi a^7/525$. It is assumed [14] that the cavity is filled with liquid of sufficiently high viscosity so that $\rho PG_0/\nu ABC \sim \varepsilon$, where G_0 and T_0 are the initial values of the angular momentum and kinetic energy of the body.

The rapid motion of a heavy rigid body about a fixed point was investigated in [8], and the rapid rotation of a heavy rigid body with a fixed point in a weakly resisting medium in [10].

Using equations derived by Chernous'ko [14], the stabilizing effect of a viscous liquid in a cavity on the rotation of a top about a given axis was studied in [12] for an arbitrary tensor P. The rapid rotation of a symmetrical top with a liquid in a gravitational field, and the possibility of the damping of nutational oscillations by a viscous liquid filling a cavity in the rotor or in the gyroscope frames was investigated in [5, 6].

We study the solution of system (1), (4) for small ε over a long time interval $t \sim 1/\varepsilon$. We solve the problem by the method of averaging [2, 11]. We average over the Euler–Poinsot motion by the method of [13, 14] for nonresonance cases.

Let us consider unperturbed motion ($\varepsilon = 0$) when the moment of the applied forces is zero. In this case the rotation of a rigid body is Euler–Poinsot motion. The quantities G, δ, λ, and T become constant, and θ, φ, and ψ are certain functions of the time t. The quantities G, δ, λ, and T are slow variables in the perturbed motion, while the Eulerian angles θ, φ, and ψ are fast variables.

For definiteness we assume that $A > B > C$, and consider motion under the condition $2TA \geq G^2 \geq 2TB$, which corresponds to paths of the angular momentum vector enclosing the z_1 axis [9]. We introduce the quantity

$$k^2 = \frac{(B-C)(2TA-G^2)}{(A-B)(G^2-2TC)} \quad (0 \leqslant k \leqslant 1), \tag{6}$$

which is a constant for unperturbed motion – the modulus of elliptic functions [9] characterizing the motion of the end of the angular momentum vector in the fixed system.

To construct the average system of the first approximation we substitute the solution of the unperturbed Euler–Poinsot motion [9] into the right-hand sides of Eqs. (1) and (4), and average over ψ and then over the time t, taking account of the dependence of θ and φ on t. The previous notation is preserved for the slow averaged variables. As a result we obtain

$$\lambda^{\cdot} = \frac{\pi mga_1}{2G^2K(k)}\sqrt{\frac{A(G^2-2TC)}{A-C}}; \quad \delta^{\cdot} = 0; \quad G^{\cdot} = -\frac{G}{A(B-C)+C(A-B)k^2}$$
$$\times\left\{I_{22}(A-C)\left[1-\frac{E(k)}{K(k)}\right]+I_{33}(A-B)\left[k^2-1+\frac{E(k)}{K(k)}\right]+I_{11}(B-C)\right.$$
$$\left.\times\frac{E(k)}{K(k)}\right\}; \quad T^{\cdot} = \frac{\rho PG^4(A-C)^2(B-C)(A-B)\,[B(A+C-B)+2AC]}{6\nu A^2B^2C^2\,[A(B-C)+C(A-B)k^2]^2}$$
$$\times\left\{(1-\varkappa)(1-k^2)-[(1-\varkappa)+(1+\varkappa)k^2]\frac{E(k)}{K(k)}\right\}-\frac{2T}{A(B-C)+C(A-B)k^2}$$
$$\times\left(I_{22}(A-C)\left[1-\frac{E(k)}{K(k)}\right]+I_{33}(A-B)\left[k^2-1+\frac{E(k)}{K(k)}\right]\right.$$
$$+\frac{(A-B)(A-C)(B-C)}{B-C+(A-B)k^2}\left\{\frac{I_{33}}{C}\left[k^2-1+\frac{E(k)}{K(k)}\right]+\right.$$

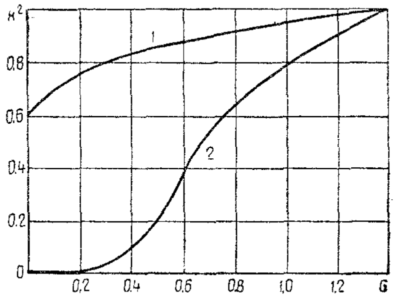

Fig. 2

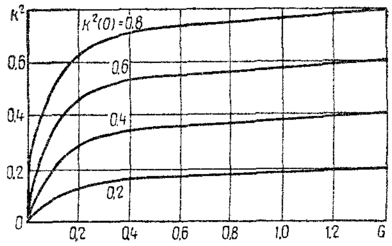

Fig. 3

$$+\frac{I_{22}}{B}(1-k^2)\left[1-\frac{E(k)}{K(k)}\right]\Bigg|+\frac{I_{11}(B-C)[A(B-C)+C(A-B)k^2]}{B-C+(A-B)k^2}\frac{E(k)}{K(k)}\Bigg). \tag{7}$$

Here K(k) and E(k) are complete elliptic integrals of the first and second kind;

$$\varkappa=\frac{3B[(A^2+C^2)-B(A+C)]}{(A-C)[B(A+C-B)+2AC]}.$$

It follows from Eq. (7) that the presence of a cavity containing a viscous liquid and a resisting medium leads to the evolution of the kinetic energy T of the body and the magnitude of the angular momentum G. It is clear that in the first approximation the viscous liquid in the cavity and the resistance of the external medium affect a change in T. The evolution of the magnitude of G occurs only under the action of the resisting force; only the diagonal components I_{ii} of the dissipative torque matrix enter the equations. Terms containing the nondiagonal elements I_{ij} $(i \neq j)$ drop out in the averaging.

The expression in curly brackets on the right-hand side of Eq. (7) for G is positive for $A > B > C$, since $(1-k^2)K \leq E \leq K$ [4]. Each coefficient in I_{ii} is a nonnegative function of k^2, and all of them cannot vanish simultaneously. Therefore, $dG/dt < 0$, i.e., G rigorously decreases for any $k^2 \in [0, 1]$.

Equation (7) for T contains terms characterizing the effect of the viscous liquid in the cavity and the resistance of the medium. According to [14] the term resulting from the effect of the liquid in the cavity is negative. Each term of the expression in curly brackets in the equation for T characterizing the effect of the resistance of the medium is positive. Thus, the kinetic energy T also rigorously decreases.

The angular velocity λ of the angular momentum vector about the vertical depends on the effect of gravity, the resistance of the medium, and the damping effect of the viscous liquid in the cavity. In the first approximation of the method of averaging, the deviation δ of the angular momentum vector from the vertical remains constant.

As the result of a number of transformations, using (6) and the last two of Eqs. (7), we obtain the differential equation for k^2

$$\frac{dk^2}{dt}=\frac{\rho PG^2(A-C)[B(A+C-B)+2AC]}{3\nu A^2B^2C^2}\Bigg\{(1-\varkappa)(1-k^2)$$

$$-[(1-\varkappa)+(1+\varkappa)k^2]\frac{E(k)}{K(k)}\Bigg\}+\frac{2(I_{33}A-I_{11}C)}{AC}\Bigg\{(1-\varkappa_1)(1-k^2)-[(1-\varkappa_1)+(1+\varkappa_1)k^2]\frac{E(k)}{K(k)}\Bigg\}. \tag{8}$$

Here

$$\varkappa_1=\frac{2I_{22}AC-I_{11}BC-I_{33}AB}{(I_{33}A-I_{11}C)B};$$

When the inequality $2TB \geq G^2 \geq 2TC$ is satisfied, corresponding to paths of the angular momentum vector enclosing the z_3 axis, it is necessary to interchange the parameters A and C and I_{11} and I_{33} in Eqs. (7) and (8), and to replace a_1 by a_3 in Eq. (7) for $\lambda^{\cdot}$. Then Eq. (8) retains its form, but $\varkappa$ must be replaced by $-\varkappa$, and $\varkappa_1$, by $-\varkappa_1$. The angular momentum approaches zero asymptotically according to a law which can be estimated as $G \sim \exp(-\gamma t)$ $(\gamma = \text{const} > 0)$. The quantity k^2 varies according to Eq. (8). The only quasistationary point of Eq. (8) is the value $k = 0$.

We note that, in contrast with previous investigations [14], the magnitude of G varies with time. In general the equations for G and k^2 cannot be integrated, and they are quite difficult to investigate.

We integrated this system numerically by computer for the initial conditions $G(0) = 1.414$, $k^2(0) = 0.99$. The value of $k^2(0)$ corresponds to motion close to passage through the separatrix. In addition, for definiteness we take $A = 3.2$, $B = 2.6$, $C = 1.67$, which corresponds to $\varkappa = 0.112$. Figure 2 shows graphs of the functions k^2 and G obtained by numerical integration. Curves 1 and 2 correspond to $\varkappa_1 = -4.471$ ($I_{11} = 2.322$, $I_{22} = 1.31$, $I_{33} = 1.425$) and $\varkappa_1 = 3.852$ ($I_{11} = 0.919$, $I_{22} = 5.228$, $I_{33} = 1.666$). It can be seen that in the first case the magnitude of the angular momentum G decreases more rapidly than k^2, while in the second case, for the values of the parameters chosen, k^2 approaches zero more rapidly than G, i.e., the motion approaches rotation about the z_1 axis.

In addition, for the first case the rate at which k^2 and G approach zero for various initial values of k^2 ($k^2(0) = 0.8$, 0.6, 0.4, 0.2) was computed numerically.

The calculated curves are shown in Fig. 3. The rate at which k^2 and G approach zero for various values of $k^2(0)$ was calculated numerically for $\varkappa_1 = -4.471$ (corresponding to curve 1 of Fig. 2, obtained for $k^2(0) = 0.99$). Thus, for a different choice of values of $k^2(0)$ for the chosen values of the parameters of the problem, G approaches zero more rapidly than k^2.

For small k^2, which corresponds to motion close to rotation about the z_1 axis, the system of equations for G^2 and k^2 takes the form

$$\frac{dG^2}{dt} = -2\frac{G^2}{A}\left\{I_{11} + \frac{k^2][(A-C)(I_{22}A - I_{11}B) + (A-B)(I_{33}A - I_{11}C)]}{2A(B-C)}\right\};$$

$$\frac{dk^2}{dt} = -\frac{k^2}{ABC}\left\{2[C(I_{22}A - I_{11}B) + B(I_{33}A - I_{11}C)] + \frac{\rho PG^2}{\nu BC}[B(A-B) + C(A-C)]\right\}. \quad (9)$$

It should be noted that (9) is a system of nonlinear differential equations describing the evolution of ecological systems [3, 7].

From (9) a first integral is determined directly in the form

$$k^{2\alpha}\exp\left\{-2\frac{(A-C)(I_{22}A - I_{11}B) + (A-B)(I_{33}A - I_{11}C)}{A^2(B-C)}k^2\right\} = C_1G^{2\beta}\exp\left\{-\frac{\rho PG^2}{\nu AB^2C^2}[B(A-B) + C(A-C)]\right\}, \quad (10)$$

where

$$\alpha = -2\frac{I_{11}}{A};\quad \beta = -\frac{2}{ABC}[C(I_{22}A - I_{11}B) + B(I_{33}A - I_{11}C)].$$

For small G^2 and k^2 it follows from (9) that the angular momentum decreases exponentially, and k^2 decreases or increases exponentially.

According to (9), for an arbitrary G the value of G^2 decreases, and the behavior of k^2 depends on the sign of the coefficient

$$C(I_{22}A - I_{11}B) + B(I_{33}A - I_{11}C).$$

The authors thank F. L. Chernous'ko for a discussion of the work and for valuable comments.

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