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SOME PROBLEMS ON THE MOTION OF A RIGID BODY WITH INTERNAL DEGREES OF FREEDOM

D. D. Leshchenko and S. N. Sallam UDC 531.1

We consider the motion of a dynamically symmetric rigid body with a spherical cavity filled with a viscous liquid and a movable mass coupled elastically to a point on the symmetry axis and experiencing viscous friction. We study the combined effect of the viscous liquid inside the cavity and the movable mass on the motion of the dynamically symmetric rigid body.

1. We consider the motion of a dynamically symmetric rigid body with a spherical cavity filled with a viscous liquid and a movable mass m which is coupled elastically to a point O_1 on the symmetry axis and experiencing viscous friction. The origin of a Cartesian coordinate system fixed to the body is taken at the center of inertia O of the body Q^* with the mass at point O_1 and the liquid. The unit vectors e_1, e_2, e_3 of this coordinate system are taken such that e_3 is along the axis of dynamical symmetry of the body Q^*. Then the position vector of the point O is $\overline{O\rho} = \rho\overline{e}_3$ and we assume with no loss of generality that $\rho > 0$. In this coordinate system the moment of inertia tensor of the rigid body Q^* has the form diag(A, A, C) where A and C are the equatorial and axial moments of inertia. In terms of components along the axes e_1, e_2, e_3 the equations of motion are [2, 3]:

$$Ap^{\cdot} + (C - A)\,qr = Fqr + Br^4p + \frac{\beta P}{\nu A^2}\,C\,(A - C)\,pr^2;$$

$$Aq^{\cdot} + (A - C)\,pr = -Fpr + Br^4q + \frac{\beta P}{\nu A^2}\,C\,(A - C)\,qr^2;$$

$$Cr^{\cdot} = -AC^{-1}Br^3(p^2 + q^2) + \frac{\beta P}{\nu A}(C - A)\,r\,(p^2 + q^2). \tag{1.1}$$

Here p, q, r are the components of the absolute angular velocity ω, β is the density of the liquid, and ν is the kinematic viscosity. The constant tensor $P = \|P_{ij}\|$ depends on the shape of the cavity and is written in the form $P_{ij} = P\delta_{ij}$, where δ_{ij} is the Kronecker delta and $P > 0$. For a spherical cavity of radius a we have $P = 8\pi a^7/525$ [2].

The basic assumption of our treatment is that the Reynolds number $R = l^2T_*^{-1}\nu^{-1}$ is small. Here l is the linear dimension of the cavity and T_* is the time scale of the motion and is inversely proportional to the characteristic angular velocity ω. Following [2], we use l and T_* as scales of length and time. In this case the viscosity is a large parameter: $\nu^{-1} \ll 1$.

Odessa Institute of Low-Temperature Technology and Power Engineering. Translated from Prikladnaya Mekhanika, Vol. 28, No. 8, pp. 58-63, August, 1992. Original article submitted March 11, 1991.

 1063-7095/92/2808-0524$12.50

To shorten the notation we put in (1.1)

$$F = m\rho^2\Omega^{-2}CA^{-3}[A^2(p^2+q^2)+C^2r^2];$$
$$B = m\rho^2\lambda\Omega^{-4}C^3(A-C)A^{-4}, \tag{1.2}$$

where $\lambda = \delta/m$, $\Omega^2 = c/m$; and δ and c are constants describing the coupling between the point and the body and play the role of the viscosity and stiffness, respectively.

We consider the case when the coupling constants are large [3]

$$\Omega^2 \gg \lambda\omega \gg \omega^2 \quad (\omega = |\omega| \sim 1). \tag{1.3}$$

In addition we assume that the motion of the rigid body generated by free vibrations of the mass m can be neglected and we consider only its driven motion relative to the body.

Multiplying the three equations of (1.1) by Ap, Aq, and Cr, respectively, and adding them together, we obtain a first integral of the system (1.1)

$$G^2 = A^2(p^2+q^2)+C^2r^2 = \text{const}, \tag{1.4}$$

which expresses the constancy (in the approximation considered here) of the magnitude of the angular momentum vector G of the rigid body Q* with solidified liquid about the point O.

Taking the time derivative of the kinetic energy

$$H = \frac{1}{2}[A(p^2+q^2)+Cr^2], \tag{1.5}$$

using the equations of motion (1.1), we obtain

$$H^{\cdot} = -m\rho^2\lambda\Omega^{-4}C^2A^{-4}(A-C)^2(p^2+q^2)r^4 - \frac{\beta P}{\nu A^2}(A-C)^2r^2(p^2+q^2) \leqslant 0, \tag{1.6}$$

As expected, the kinetic energy is a decreasing function of time.

We introduce the angles θ and φ determining the orientation of the vector G relative to the rigid body

$$Ap = G\sin\theta\cos\varphi, \quad Aq = G\sin\theta\sin\varphi, \quad Cr = G\cos\theta. \tag{1.7}$$

Here θ is the mutation angle and φ is the precession angle.

We transform (1.1) to the variables (1.7), taking into account the constancy of G. Solving the resulting equations for the derivatives $\varphi^{\cdot}$ and $\theta^{\cdot}$ and substituting (1.2), we obtain

$$\theta^{\cdot} = \alpha\sin\theta\cos^3\theta + \eta\sin\theta\cos\theta, \quad \varphi^{\cdot} = \gamma\cos\theta. \tag{1.8}$$

Here

$$\alpha = m\rho^2\lambda\Omega^{-4}C^{-1}A^{-5}(A-C)G^4 = \text{const},$$
$$\eta = \beta PG^2\nu^{-1}A^{-3}C^{-1}(A-C) = \text{const};$$
$$\gamma = G(C-A-m\rho^2\Omega^{-2}CA^{-3}G^2)A^{-1}C^{-1} = \text{const}. \tag{1.9}$$

The quantities α, η, γ have the dimensions of angular velocity and are constants of the motion. We note that in the case of complete symmetry (A = C) (1.8) and (1.9) become

$$\omega = \text{const}, \quad \theta = \text{const}, \quad \varphi^{\cdot} = -m\rho^2\Omega^{-2}A^{-1}\cos\theta = \text{const}, \tag{1.10}$$

which corresponds to uniform rotation of the vector ω about the axis e. In this case of complete symmetry e is a unit vector along $\bar{\rho}$, i.e., $\bar{\rho} = \rho e$. The position of the vector ω in the coordinate system rigidly fixed to the body is given by the angle θ with the vector e and the angle φ between the projection of ω onto the plane perpendicular to e and a fixed direction in this plane. The equations (1.10) are written in terms of these variables.

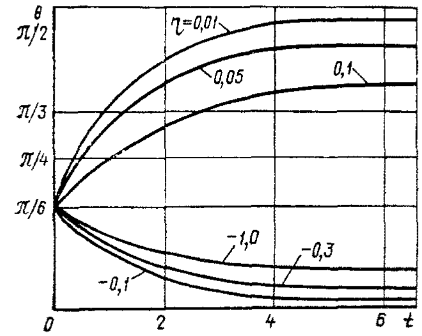

Fig. 1

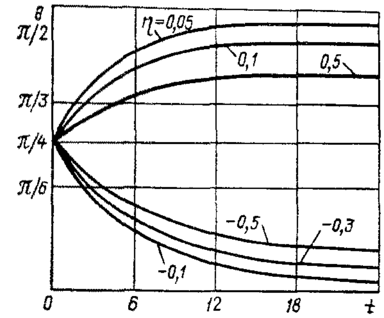

Fig. 2

In the limit $\lambda \to 0$, $\rho \to 0$, $\alpha \to 0$ (1.8) and (1.9) reduce to the results of [2] for the motion of a rigid body with a cavity filled with a viscous liquid. When $\nu \to \infty$, $\eta \to 0$ these expressions reduce to the results of [3] for the motion of a rigid body with a movable mass.

Integrating the first equation of (1.8), we find for a dynamically symmetric body

$$\frac{\alpha}{2\eta(\alpha+\eta)} \ln(\alpha + \eta \sec^2\theta) + \frac{1}{\alpha+\eta} \ln|\operatorname{tg}\theta| = t + \text{const}, \tag{1.11}$$

where α and η are given by (1.9). Applying the initial condition $\theta(t_0) = \theta_0$, we obtain from (1.11)

$$(1 + \sigma \sec^2\theta)(\operatorname{tg}^2\theta)^\sigma = (1 + \sigma \sec^2\theta_0)(\operatorname{tg}^2\theta_0)^\sigma \exp[2\eta(1+\sigma)t]. \tag{1.12}$$

Here we have assumed with no loss of generality that θ lies in the first quadrant (if this is not the case then we can reverse the directions of the axes). In (1.12) we have put

$$\sigma = \frac{\eta}{\alpha} = \frac{A^2\beta P\Omega^4}{m\nu\lambda\rho^2G^2}. \tag{1.13}$$

The result (1.12) is an implicit relation between θ and t. According to (1.9), the sign of η is determined by the sign of $A - C$. It is evident from (1.12) that when $A > C$ (prolate body) the angle θ increases monotonically with t and approaches $\pi/2$ as $t \to \infty$. The final motion will be a rotation about an axis perpendicular to the axis of dynamical symmetry. If $A < C$ (oblate body) then θ decreases monotonically and approaches zero as $t \to \infty$. The final motion in this case will be a rotation about the axis of dynamical symmetry.

Therefore the direction of the angular momentum vector G in the coordinate system fixed to the body approaches a steady state along the axis corresponding to the largest moment of inertia. The time dependence of the angle φ is found from the second equation of (1.8).

Figures 1 and 2 show graphs of the nutation angle $\theta(t)$ for the initial values $\theta_0 = \pi/6$ and $\pi/4$ and different values of the parameter η, which are indicated on the curves. In the approximation considered here the quantity $\theta(t)$ approaches a right angle and zero as $\eta \to \pm 0$, respectively. This supports the above conclusions on the motion of the body.

The angular momentum G_0 of the entire system, which is conserved in a nonmoving coordinate system, differs from the vector G by the angular momentum k of the mass m and by the gyrostatic angular momentum L. For the conditions (1.3) the vector k is of order $O(\Omega^{-2})$ [3]. The vector L was calculated in [2] and is of order $O(\nu^{-1})$. Therefore our results show that in the presence of internal dissipation the motion of the system approaches a steady rotation about the axis of largest moment of inertia as $t \to \infty$. This qualitative conclusion is well-known (see [2, 3]) and follows from energy considerations. Indeed, the kinetic energy of the rigid body Q*, using (1.7), is

$$H = \frac{1}{2}[A(p^2+q^2) + Cr^2] = 1/2A^{-1}G^2[1 + (A - C)C^{-1}\cos^2\theta]. \tag{1.14}$$

We see that the minimum value of H corresponds to $\theta = \pi/2$ for $A > C$ and to $\theta = 0$ for $A < C$, which agrees with the results obtained above. In the case considered here the movable mass and the cavity with the liquid inside the body only slightly change its angular momentum, but can lead to significant energy dissipation.

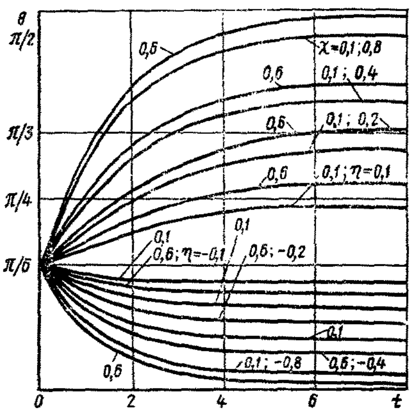

Fig. 3

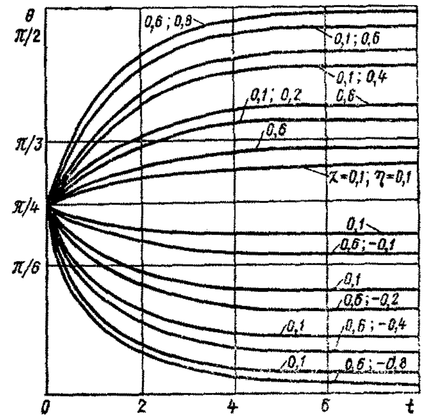

Fig. 4

2. We consider the motion of a dynamically symmetric rigid body with a spherical cavity filled with a viscous liquid and a movable point mass m coupled elastically with the body at a point O_1 on the symmetry axis and experiencing quadratic friction with coefficient μ. The origin of a Cartesian coordinate system fixed to the body is taken at the center of inertia O of the system Q^* consisting of the body with the liquid and the point mass at O_1. The unit vectors of this coordinate system e_1, e_2, e_3 are such that e_3 is along the axis of dynamical symmetry of the system. Then the position vector $\overline{\rho}$ of point O_1 is $\overline{\rho} = \rho e_3$ and we assume the case $\rho > 0$. In this coordinate system the moment of inertia tensor of the system Q^* has the form diag(A, A, C), where A and C are the equatorial and axial moments of inertia, respectively. Taking components along e_1, e_2, e_3, the equations of motion are [1, 2]:

$$Ap^{\cdot} + (C - A)\,qr = Nqr + Spr^6 + \frac{\beta P}{\nu A^2} C\,(A - C)\,pr^2;$$

$$Aq^{\cdot} + (A - C)\,pr = -Npr + Sqr^6 + \frac{\beta P}{\nu A^2} C\,(A - C)\,qr^2;$$

$$Cr^{\cdot} = -SAC^{-1}r^5\,(p^2 + q^2) + \frac{\beta P}{\nu A}(C - A)\,r\,(p^2 + q^2). \tag{2.1}$$

Here p, q, r are the components of the absolute angular velocity ω, β is the density of the liquid, and ν is the kinematic viscosity.

For a spherical cavity of radius a we have $P = 8\pi a^7/525$ [2]. The Reynolds number is assumed to be small, i.e., the viscosity of the liquid is large.

To shorten the notation in (2.1) we introduce the notation

$$N = m\rho^2\Omega^{-2}CA^{-3}G^2, \quad G^2 = A^2\,(p^2 + q^2) + C^2r^2;$$
$$S = m\rho^3\Lambda\Omega^{-3}C^4A^{-4}d|d|\,(p^2 + q^2)^{1/2}, \quad d = 1 - CA^{-1}, \tag{2.2}$$

where $\Omega_2 = c/m$, c is the stiffness of the elastic coupling, and $\lambda_1 = \mu/m = \lambda\Omega^3$, $\Omega \gg \omega$.

We consider the case when the coupling coefficients λ_1 and Ω are such that the "free" motion of the point mass m caused by an initial deviation damps out rapidly compared to the period of rotation of the body [1]. Then the motion of the body will be close to the Euler–Poisson motion and the relative vibrations of the point mass driven by the motion of the body will be small.

Multiplying the three equations of (2.1) by Ap, Aq, and Cr, respectively, and adding these equations together, we find a first integral of the motion: the magnitude of the angular momentum $G = |\overline{G}|$

$$G = \text{const}. \tag{2.3}$$

To determine the quantity ω we use the following method [1-3]. We write the components of the vector G along the principal central axes of inertia as follows:

$$Ap = G \sin\theta \cos\varphi, \quad Aq = G \sin\theta \sin\varphi, \quad Cr = G \cos\theta. \tag{2.4}$$

Here θ is the nutation angle and φ is the precession angle. Since G = const, differentiating (2.4) and using the equations of motion (2.1) and the expressions (2.2), we obtain the following differential equations for the spherical angles θ and φ

$$\varphi^{\cdot} = \varkappa \cos\theta, \quad \theta^{\cdot} = \eta \sin\theta \cos\theta + \xi \sin\theta \,|\sin\theta|\, \cos^5\theta. \tag{2.5}$$

The coefficients $\varkappa$, η, ξ in (2.5) are constant and equal to

$$\varkappa = -GC^{-1}(d + m\rho^2\Omega^{-2}CA^{-4}G^2);$$
$$\eta = \beta P(A - C)G^2\nu^{-1}A^{-3}C^{-1}, \quad \xi = m\rho^3\Lambda\Omega^{-3}C^{-2}A^{-6}d\,|d|\,G^7. \tag{2.6}$$

In the special case of spherical symmetry (A = C, d = 0) it follows from (2.6) that the constants η and ξ are equal to zero and (2.5) can be integrated explicitly $\theta = \theta_0$, $\varphi = \varkappa t \cos\theta_0 + \varphi_0$, θ_0, $\varphi_0 = \text{const}$.

We consider now the general case $\eta \neq 0$, $\xi \neq 0$. Equation (2.5) for the angle θ can be written in the form

$$\int_{\theta_0}^{\theta} \frac{d\theta}{\sin\theta\cos\theta + \chi \sin^2\theta \cos^5\theta} = \theta \int_0^t dt. \tag{2.7}$$

Here we have assumed with no loss of generality that θ lies in the first quadrant

$$\chi = \frac{\xi}{\eta} = \frac{m\rho^3\Lambda\,|d|\,\nu G^5}{\beta P \Omega^3 C A^4}.$$

The solution of (2.7) was calculated numerically for different χ and η. Figures 3 and 4 show graphs of the nutation angle $\theta(t)$ for the initial conditions $\theta_0 = \pi/6$, $\pi/4$ and different values of the parameters χ and η, which are indicated on the corresponding curves. According to (2.6) the sign of the quantity η is determined by the sign of the difference A − C. It is evident from the graphs that for A > C ($\eta > 0$, prolate body) the angle θ approaches $\pi/2$ as χ and η increase. In this case it follows from (2.5) that $\dot{\varphi} \to 0$. When A < C ($\eta < 0$, oblate body) the angle θ goes to zero as χ increases and η decreases. In this case $\dot{\varphi} \to \varkappa$ = const according to (2.5)

Therefore in all cases the angular momentum of the body G in a coordinate system fixed to the body approaches the axis of largest moment of inertia.

The authors thank F. L. Chernous'ko and L. D. Akulenko for useful discussions.

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