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Optimal Rotation Deceleration of a Dynamically Symmetric Body with Movable Mass in a Resistant Medium

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Abstract—A minimum-time problem on deceleration of rotation of a free rigid body is studied. The body is assumed to contain a viscous–elastic element, which is modeled as a movable point mass attached to the body via a damper. In addition, the body is subjected to a retarding torque generated by linear medium resistance forces. In an undeformed state, the body is assumed to be dynamically symmetric, with the mass being located on the symmetry axis. An optimal control law for deceleration of rotation of the body is synthesized, and the corresponding time and phase trajectories are determined.

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INTRODUCTION

The progress in studies of dynamics and control of rigid body motion is associated with taking into account the fact that bodies are nor absolutely rigid and that the ideal models are just approximations of real ones. The effect of imperfections can be revealed by means of asymptotic methods of nonlinear mechanics (singular perturbations, averaging, and the like). This effect manifests itself in additional perturbing moments in the Euler equations of motion for some fictitious rigid body. Much attention was paid to the analysis of passive motion of rigid bodies carrying a movable mass elastically attached to the body, which moves in a resistant medium, in the presence of viscous friction [1–5]. Rotation control of "quasi-rigid" bodies by means of concentrated torques applied to the body received much less attention. A class of systems with smooth controls was identified for which singular perturbation methods do not result in accumulation of errors of the "boundary layer" type [6–8].

Below we consider a minimum-time problem of deceleration of rotations of a dynamically symmetric rigid body containing a viscous–elastic element, which is modeled as a concentrated mass connected to the body via a damper at a point on the symmetry axis. In addition, the rigid body is subjected to a retarding torque generated by linear medium resistance forces. The rotation is controlled by a torque of bounded magnitude. The model under consideration is a generalization of models studied earlier in [6–8], where the minimum-time problem on stabilization of a dynamically symmetric rigid body with a movable mass attached to the body via a viscous–elastic connection was solved. In [7], minimum-time optimal deceleration of rotation of a dynamically symmetric body with a spherical cavity completely filled by highly viscous liquid was studied for low Reynolds number. In addition, the body contains a viscous–elastic element modeled as a concentrated mass connected to the body via a damper at a point on the symmetry axis. It is shown in monograph [8] that the functional Schwartz inequality turns out very useful in synthesizing control laws for deceleration of "quasi-rigid" bodies. Approximate solutions of perturbed minimum-time problems on rotation deceleration of rigid bodies relative to the center of mass, including objects with internal degrees of freedom, which have applications in dynamics of space- and aircrafts, are obtained. A number of mechanical models are invariant with respect to the angular momentum. Deceleration of bodies with cavities filled by viscous liquid is studied. The cases of axially symmetric and asymmetric in the unperturbed states bodies that have spherical cavities filled by liquid of high viscosity are considered for low Reynolds numbers. Deceleration of perturbed rotation of an almost spherically symmetric rigid body under the action of a moment of linear forces of medium resistance directed against the angular velocity vector is analyzed. The minimum-time problem on deceleration of perturbed rotation of an asymmetric rigid body under the action of the moment of linear friction forces is solved.

1. OPTIMAL CONTROL PROBLEM STATEMENT

By means of the approach suggested in [1, 8], equations of controlled rotations projected onto the axes of the body frame (the Euler equations) can be written in the form [1, 3, 4, 8]

$$A_1\dot{p} + (A_3 - A_1)qr = M_p + FG^2qr + Dr^4p - \chi A_1 p,$$

$$A_1\dot{q} + (A_1 - A_3)pr = M_q - FG^2pr + Dr^4q - \chi A_1 q, \qquad (1.1)$$

$$A_3\dot{r} = M_r - A_1A_3^{-1}Dr^3\left(p^2 + q^2\right) - \chi A_3 r.$$

Here, p, q, r are projections of the vector of absolute angular velocity $\boldsymbol{\omega}$ onto the axes of the body frame, $\mathbf{J} = \mathrm{diag}(A_1, A_1, A_3)$ is the inertia tensor of the unperturbed body, $M_{p,q,r}$ are projections of torque $\mathbf{M}$, and $\mathbf{G} = J\boldsymbol{\omega}$ is the angular momentum. The magnitude of the angular momentum is given by

$$G = |\mathbf{G}| = \left[A_1^2\omega_\perp^2 + A_3^2r^2\right]^{1/2}, \quad \omega_\perp^2 = p^2 + q^2.$$

To simplify the problem, a structural constraint is introduced into system (1.1): it is assumed that the diagonal tensor of the moment of external resistance forces is proportional to the tensor of the moment of the inertia forces; i.e., the moment of the dissipation forces is proportional to the angular momentum:

$$\mathbf{M}^r = -\chi J\boldsymbol{\omega}, \qquad (1.2)$$

where χ is a constant coefficient of proportionality depending on the medium. Resistance acting on the body is presented by a pair of forces applied to the body. Projections of the moment of this pair onto the principal axes of body inertia are given by $\chi A_1 p$, $\chi A_1 q$, and $\chi A_3 r$ [3, 4]. Such an assumption is not contradictive.

It is also assumed that admissible values of the control torque $\mathbf{M}$ belong to the ball [8]

$$\mathbf{M}^u = b\mathbf{u}, \quad |\mathbf{u}| \le 1; \quad b = b(t, \mathbf{G}), \quad 0 < b_* \le b < b^* < \infty, \qquad (1.3)$$

where b is a scalar function bounded in the considered domain of arguments t and $\mathbf{G}$ according to (1.3). This domain is either defined a priori or estimated by the initial data for $\mathbf{G}$, $\mathbf{G}(t_0) = \mathbf{G}^0$.

Parameters F and D introduced in (1.1) are expressed in terms of the system parameters as follows:

$$F = m\rho^2\Omega^{-2}A_3A_1^{-3}, \quad D = m\rho^2\lambda\Omega^{-4}A_3^3(A_1 - A_3)A_1^{-4}. \qquad (1.4)$$

Coefficients D and F in (1.4) characterize perturbing moments due to viscous–elastic element, m is the mass of the movable concentrated mass, and ρ is the distance from the center of mass of the undeformed system to the attachment point, which, by assumption, lies on the axis of dynamic symmetry of the body. Constants $\Omega^2 = c/m$ and $\lambda = \delta/m$ are vibration frequency and rate of damping, respectively; c is rigidity (elasticity coefficient); and δ is the damper viscosity coefficient. We consider the case of a strong damper, where the connection coefficients are large in the following sense [1]:

$$\Omega^2 \gg \lambda\omega \gg \omega^2. \qquad (1.5)$$

Strong inequalities (1.5) make it possible to introduce a small parameter in (1.4) and assume that the specified perturbing moments are small, which allows us to use asymptotic averaging methods. In addition, conditions (1.5) allow us to neglect free boundary layer vibrations of the mass caused by initial deviations (owing to their rapid attenuation) and to take into account forced quasi-stationary motions caused by rotations of the body. Note that the mass m may be significant and comparable with the mass of the body.

Thus, in the quasi-static approximation, the perturbing moments due to damper elasticity and viscosity are determined by the monomials of the components of vector $\boldsymbol{\omega}$ to the fourth and fifth powers, respectively. Small retarding moment of medium resistance is linear with respect to the angular velocity of the perturbation. The mathematical model of controlled rotations of a quasi-rigid body is constructed in the form the Euler equations (1.1).

We pose the following minimum-time problem on rotation deceleration:

$$\boldsymbol{\omega}(T) = 0, \quad T \to \min_{\mathbf{u}}, \quad |\mathbf{u}| \le 1. \qquad (1.6)$$

It is required to synthesize an optimal control law $u = u(t, \boldsymbol{\omega})$, construct the corresponding trajectory $\boldsymbol{\omega}(t, t_0, \boldsymbol{\omega}^0)$, and find time $T = T(t_0, \boldsymbol{\omega}^0)$ and the Bellman function $W = T(t, \boldsymbol{\omega}) - t$.

2. SOLUTION OF THE OPTIMAL DECELERATION PROBLEM

Note that the moment due to motion of the viscous–elastic element is an internal moment for the fictitious body, and the moment of the linear resistance forces is external. Based on the dynamic programming, the minimum-time control law has the following form [8]:

$$M_p = -b\frac{A_1 p}{G}, \quad M_q = -b\frac{A_1 q}{G}, \quad M_r = -b\frac{A_3 r}{G}, \quad b = b(t, G). \tag{2.1}$$

Here, for the sake of simplification, we assume that $b = b(t, G)$, $0 < b_1 \le b \le b_2 < \infty$. Let us multiply the first equation in (1.1) by $A_1 p$, the second equation by $A_1 q$, and the third equation by $A_3 r$ and add them together. We obtain the scalar equation to be integrated and the equation for T:

$$\dot{G} = -b(t, G) - \chi G, \quad G(t_0) = G^0, \quad G(T, t_0, G^0) = 0,$$
$$T = T\left(t_0, G^0\right), \quad W(t, G) = T(t, G) - t.$$

Assuming that $b = b(t)$, we obtain the following solution and condition for T:

$$G(t) = G^0 e^{-\chi(t - t_0)} - \int_{t_0}^{t} b(\tau) e^{-\chi(t-\tau)} d\tau, \quad G^0 = e^{-\chi t_0} \int_{t_0}^{T} b(\tau) e^{\chi\tau} d\tau, \quad T = T\left(t_0, G^0\right). \tag{2.2}$$

Here, t is the current time in the deceleration process, and T is the minimum time. For $b = \text{const}$, solution of the equation and the boundary value problem (2.2) is written as

$$G(t) = \frac{1}{\chi}\left[\left(G^0\chi + b\right)\exp(-\chi t) - b\right], \quad T = \frac{1}{\chi}\ln\left(G^0\frac{\chi}{b} + 1\right), \quad t_0 = 0. \tag{2.3}$$

Below, case (2.3) is analyzed in detail.

3. ANALYSIS OF AXIAL ROTATION FOR CONTROLLED MOTION OF THE BODY

Substitution of the known expression for G into the third equation in (1.1) results in the following elementary nonlinear equation in r:

$$\dot{r} = -r\left[bG^{-1} + \chi + A_1^{-1}A_3^{-2}Dr^2\left(G^2 - A_3^2 r^2\right)\right]. \tag{3.1}$$

After replacement of the axial component of the angular velocity vector $r = GR$, where R is an unknown function, Eq. (3.1) takes the form that admits separation of variables and trivial integration:

$$\dot{R} = -A_1^{-1}A_3^{-2}DG^4R^3\left(1 - A_3^2R^2\right). \tag{3.2}$$

The projection of the vector of angular momentum **G** onto the principal central inertia axes of the body
1 yields $A_3 r = G\cos\theta$, where θ is the nutation angle. As result, we obtain the relation $A_3 R = \cos\theta$ in unknown R. Turning to the unknown θ in (3.2), we obtain

$$\dot{\theta} = A_1^{-1}A_3^{-4}D\chi^{-4}\left[\left(G^0\chi + b\right)\exp(-\chi t) - b\right]^4\cos^3\theta\sin\theta, \quad \theta(0) = \theta^0. \tag{3.3}$$

The solution to (3.3) is written as follows:

$$\begin{aligned}
&\mathrm{tg}^2\theta\exp\left(\mathrm{tg}^2\theta\right) = \mathrm{tg}^2\theta^0\exp\left(\mathrm{tg}^2\theta_0\right)\exp\left(2K(t)\right),\\
&K(t) = A_1^{-1}A_3^{-4}D\chi^{-4}\left[\frac{1}{4}\chi^{-1}\left(G^0\chi + b\right)^4\left(1 - \exp(-4\chi t)\right)\right.\\
&\quad - \frac{4}{3}\chi^{-1}b\left(G^0\chi + b\right)^3\left(1 - \exp(-3\chi t)\right)\\
&\quad \left. + 3\chi^{-1}b^2\left(G^0\chi + b\right)^2\left(1 - \exp(-2\chi t)\right) - 4\chi^{-1}b^3\left(G^0\chi + b\right)\left(1 - \exp(-\chi t)\right) + b^4 t\right].
\end{aligned} \tag{3.4}$$

Without loss of generality, we may assume that the initial value $\theta(0) = \theta^0$ belongs to the first quadrant $(0 \le \theta^0 \le \pi/2)$. If θ^0 takes values from the specified interval, then, in the course of rotation evolution, the
1 nutation angle will remain in this interval as well, since $\theta^* = 0$ and $\theta^* = \pi/2$ are stationary points of Eq. (3.4) independent of variation of G. Equation (3.4) implicitly determines dependence of angle θ on t. The left-hand side of this equation is a monotonically growing function of $|\mathrm{tg}\theta|$, and the right-hand side is a monotone function of t. Hence, relation (3.4) defines a unique monotone function $\theta(t)$. The behavior of this function is similar to that studied in [1].

1 Let us study behavior of the nutation angle in a small half-neighborhood of the stationary point $\theta^* = 0$ of Eq. (3.3): $\theta = \delta\theta > 0$. Equation (3.3) takes the form

$$\delta\dot{\theta} = A_1^{-1} A_3^{-4} D\chi^{-4}\left[(G^0\chi + b)\exp(-\chi t) - b\right]^4 \delta\theta, \quad |\delta\theta| = |\delta\theta_0|\exp(K(t)). \tag{3.5}$$

From (3.5), it follows that variation $\delta\theta$ monotonically decreases when $A_3 > A_1$ (dynamically flattened body) and monotonically increases when $A_3 < A_1$ (dynamically prolate body), since $D > 0$ or $D < 0$, respectively.

For $A_1 \approx A_3$ and $|\delta\theta^0|$|, perturbation methods can be applied in a neighborhood of stationary points, which, in the given case, result in elementary expressions. For example, after the first iteration, we have

$$\theta(t) = \theta^0 + A_1^{-1} A_3^{-4} D\chi^{-5} \sin\theta^0 \cos^3\theta^0 K(t). \tag{3.6}$$

1 Formula (3.6) makes it possible to carry out analysis of the nutation angle variation in time for various values of system parameters and initial data.

4. ANALYSIS OF BODY ROTATION IN THE EQUATORIAL PLANE

Consider now variation of the equatorial components of variables p and q in accordance with the first two equations in (1.1). Let us introduce variable $N = A_1\omega_\perp$, which is the absolute value of the above-specified components characterizing these rotations. Multiplying the first equation in (1.1) by $A_1 p N^{-1}$ and the second equation by $A_1 q N^{-1}$ and adding them together, we obtain the following linear homogeneous equation in N:

$$\dot{N} = -d(t)N, \quad d(t) = \frac{b(t)}{G(t)} - A_1^{-1} D r^4(t) + \chi, \quad d(t) > 0. \tag{4.1}$$

After integration, we have

$$N(t) = N^0 \exp\left[-\int_{t_0}^{t} d(\tau)d\tau\right], \quad N^0 \equiv A_1((p^0)^2 + (q^0)^2)^{1/2}. \tag{4.2}$$

On the other hand, the magnitude of the angular momentum squared can be represented as

$$G^2 = N^2 + A_3^2 r^2.$$

From this equation, we obtain the expression for N

$$N = \left(G^2 - A_3^2 r^2\right)^{1/2}$$

or, taking into account relation $A_3 r = G\cos\theta$,

$$N = G\sin\theta. \tag{4.3}$$

For $b = \text{const}$, with regard to (2.3), we have

$$N = \frac{1}{\chi}\left[\left(G^0\chi + b\right)\exp(-\chi t) - b\right]\sin\theta. \tag{4.4}$$

Numerical analysis of variation of angle θ is presented in Section 5.

Using known expressions for $G(t)$ and $r(t)$, we reduce the equations in p and q to linear equations with time-varying coefficients and certain symmetry. These equations contain only gyroscopic and dissipative terms with coefficients $g(t)$ and $d(t)$, respectively,

$$\dot{\mathbf{N}} = -d(t)\mathbf{N} + g(t)I\mathbf{N}, \quad \mathbf{N} = (A_1 p, A_1 q)^T, \quad g(t) = A_1^{-1} r(t)\left(A_1 - A_3 + FG^2(t)\right). \tag{4.5}$$

Here, I is the symplectic matrix, and coefficient d is defined in (4.1). The gyroscopic coefficient $g(t)$ coincides with that in the problem of motion of a body with a spherical cavity filled with liquid of high viscosity and viscous–elastic element [7].

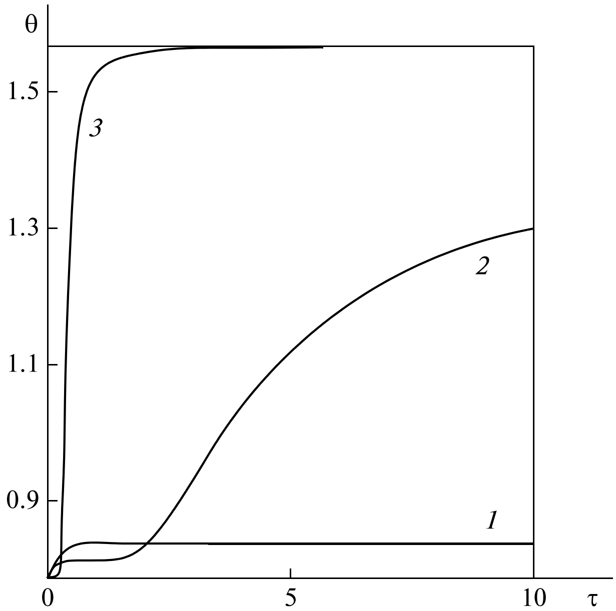

Fig. 1.

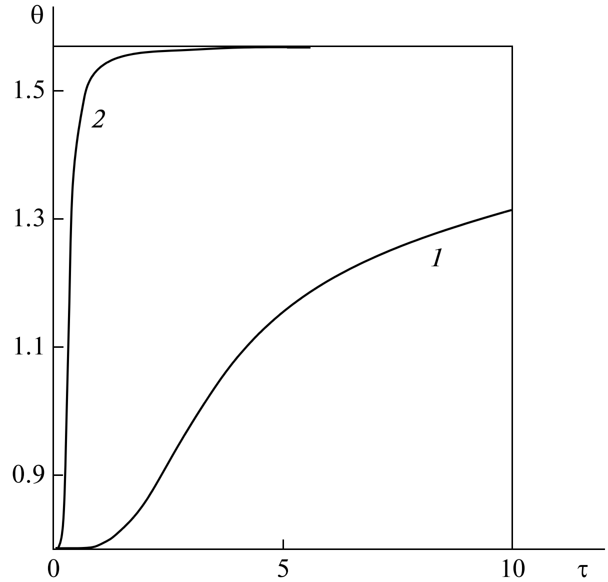

Fig. 2.

Equation (4.1) for **N** is integrated explicitly. Setting $\mathbf{N} = N\mathbf{n}$, where $\mathbf{n} = \mathbf{n}(t)$ is the unit vector of **N**, we obtain equation $\mathbf{n}' = g(t)I\mathbf{n}$ in the unknown **n**. The initial value $\mathbf{n}(0) = \mathbf{n}^0$, $|\mathbf{n}^0| = 1$, is specified by the condition $\mathbf{N}^0 = N^0\mathbf{n}^0$. Note that $|\mathbf{n}(t)| \equiv 1$ for all $t \in [0, T_0]$. Let us introduce argument σ such that $\mathbf{n}^0 = I\mathbf{n}$. We have

$$\mathbf{n}^0(t) = \Pi(\sigma)\mathbf{n}^0, \quad \sigma = \int_0^t g(\tau)\,d\tau, \quad \Pi(\sigma) = \begin{bmatrix} \cos\sigma & \sin\sigma \\ -\sin\sigma & \cos\sigma \end{bmatrix}, \tag{4.6}$$

where $\Pi(\sigma)$ is the matrix of rotation (of the initial vector $\mathbf{n}^0$) through angle σ.

Thus, the precession rotation of a quasi-rigid body relative to an axis in the equatorial plane is completely determined by (4.2) and (4.6).

5. NUMERICAL RESULTS AND ANALYSIS

1 Consider again the problem of determining the nutation angle $\theta(t)$ in the particular case of $b = \text{const}$ in accordance with (3.3). First, let us write Eq. (3.3) in the dimensionless form. To this end, we introduce the notation

$$\tau = \chi t, \quad k^* = \frac{|D|^{1/4} k}{A_1^{1/4} A_3 \chi^{1/4}}, \quad G^{0*} = \frac{G^0 |D|^{1/4}}{A_1^{1/4} A_3 \chi^{1/4}}, \quad k = b\chi^{-1}. \tag{5.1}$$

1 After these transformations, we obtain the following equation for the nutation angle θ:

$$\frac{d\theta}{d\tau} = \operatorname{sign}(D)\left[\left(G^{0*} + k^*\right)\exp(-\tau) - k^*\right]^4 \sin\theta\cos^3\theta. \tag{5.2}$$

Equation (5.2) was integrated numerically for arbitrary values of G^{0*}, k^*, and initial angle $\theta^0 = \pi/4$ rad.
1 Plots showing variation of the nutation angle θ are presented in Figs. 1–3. Figures 1 and 2 correspond to the dynamically prolate body, and Fig. 3, to the dynamically flattened body.

Figure 1 corresponds to the initial value of the dimensionless angular momentum equal to $G^{0*} = 1$. Curves *1–3* are constructed for different values of $k^* = 0.1$, 1, and 10. The calculation showed that, for
1 the dynamically prolate body ($A_1 > A_3$), the nutation angle tends to its limit value $\pi/2$ rad. Figure 1 presents dimensionless time interval $\tau \le 10$. It can be seen from curve *3* that, upon considerable effect of

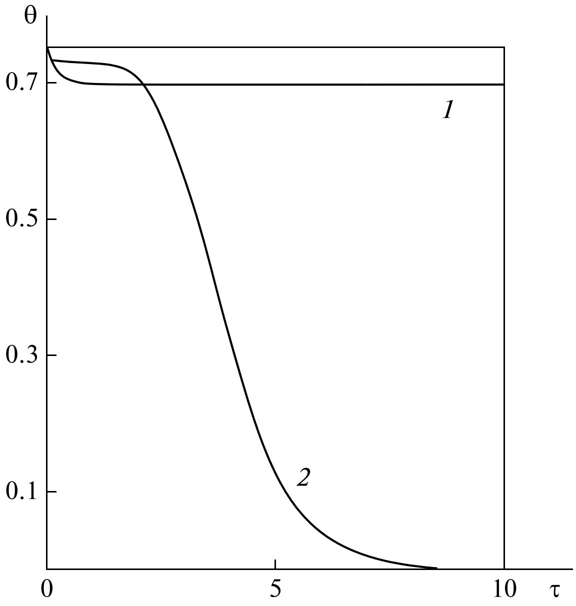

Fig. 3.

dimensionless coefficient of the control torque ($k^* = 10$), the nutation angle rapidly reaches its limit value. Note that the body manages to decelerate, since the optimal time in this case is one order of magnitude less than the calculation time. The smaller the value of k^*, the slower the rate of convergence of the symmetry axis to its limit position. However, in all cases considered, the calculation time is greater than the time required for the body to decelerate.

Curves *1* and *2* in Fig. 2 correspond to $k^* = 1$ and 10, respectively, for $G^{0*} = 0.1$. The curve corresponding to $k^* = 0.1$ is not shown in the figure, since the nutation angle almost does not change during the calculation time. As can be seen, the greater the value of k^*, the faster the rotation axis of the body reaches its stable limit position. A similar behavior of function $\theta(t)$ was observed in [6, 7].

The variation of the nutation angle for the dynamically flattened body ($A_1 < A_3$) was studied numerically. Figure 3 shows plots of function $\theta(t)$ for $G^{0*} = 1$. Curve *1* corresponds to $k^* = 0.1$, and curve *2*, to $k^* = 1$. As can be seen from curve *2*, the rotation axis of the dynamically flattened body tends to its limit stable position: $\theta \to 0$ rad. The convergence rate depends on the value of the dimensionless coefficient of the control torque. The greater the value of this coefficient, the faster the body axis reaches its limit position. Note that the optimal time in this case considerably reduces. The case of $k^* = 10$ is not presented in Fig. 3, since the body decelerates almost instantaneously in this case.

Calculations showed that the behavior of function $\theta(t)$ in this problem coincides with the behavior of the nutation angle for the rigid body with movable masses inside the body [1]. Thus, direction of the angular momentum vector **G** in the body frame tends to its stationary position, namely, to the direction of the axes corresponding to the greatest inertia moments.

CONCLUSIONS

A minimum-time control problem on deceleration of rotations of a dynamically symmetric body with a viscous–elastic element in a resistant medium has been analytically and numerically studied. The control law, minimum time (Bellman function), and the nutation angle have been determined by using the asymptotic approach. Qualitative features of the optimal motion have been established.

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