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OPTIMAL CONTROL

Optimal Rotation Deceleration of a Dynamically Asymmetric Body in a Resistant Medium

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Abstract—A minimum-time problem on deceleration of rotation of a free body is studied. The body is subject to a retarding torque of viscous friction. The body is assumed to be dynamically asymmetric. An optimal control law for deceleration of rotation of the body is synthesized, and the corresponding time and phase trajectories are determined.

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INTRODUCTION

Analysis of hybrid systems, i.e., objects containing distributed and lumped elements, is of interest from the both theoretical and applied standpoints. For systems containing "quasi-rigid" bodies, various approaches have been developed and significant results have been obtained. Motion of the "quasi-rigid" bodies is assumed to be similar to that of absolutely rigid bodies. The effect of imperfections can be revealed by means of asymptotic methods of nonlinear mechanics (singular perturbations, averaging, etc.). This effect manifests itself in additional terms in the Euler equations of motion for some fictitious rigid body. Much more attention was paid to the analysis of passive motion of rigid bodies in a resistant medium [1–3]. Rotation control of "quasi-rigid" bodies by means of concentrated torques, which is an important applied problem, received much less attention.

Below we consider a minimum-time problem of deceleration of rotations of a dynamically asymmetric body. The rigid body is subjected to a retarding torque generated by linear medium resistance forces. The rotation is controlled by a torque of bounded magnitude. It is shown in monograph [4] that the Schwartz inequality turns out very useful in synthesizing control laws of deceleration of "quasi-rigid" bodies. Approximate solutions of perturbed minimum-time problems on rotation deceleration of rigid bodies, including objects with internal degrees of freedom, which have applications in dynamics of space- and air-crafts, are obtained. A number of mechanical models are invariant with respect to the angular momentum. Deceleration of bodies with cavities filled by viscous liquid is studied. Deceleration of perturbed rotation of an almost spherically symmetric rigid body under the action of a moment of linear forces of medium resistance directed against the angular velocity vector is analyzed.

1. OPTIMAL CONTROL PROBLEM STATEMENT

By means of the approach suggested in [4], equations of controlled rotations projected onto the axes of the body frame (the Euler equations) can be written in the form [1–4]

$$J\dot{\boldsymbol{\omega}} + [\boldsymbol{\omega} \times J\boldsymbol{\omega}] = \mathbf{M} - \lambda J\boldsymbol{\omega}. \tag{1.1}$$

Here, $\boldsymbol{\omega} = (p, q, r)$ is the vector of absolute angular velocity, $J = \mathrm{diag}(A, B, C)$ is the inertia tensor of the body, $\mathbf{M}$ is the control torque, and $\mathbf{L} = J\boldsymbol{\omega}$ is the angular momentum. The magnitude of the angular momentum is given by

$$G = |\mathbf{L}| = (A^2p^2 + B^2q^2 + C^2r^2)^{1/2}.$$

To simplify the problem, a structural constraint is introduced into system (1.1): it is assumed that the diagonal tensor of the moment of linear resistance forces is proportional to the tensor of the moment of the inertia forces; i.e., the moment of the dissipation forces is proportional to the angular momentum. It is also assumed that admissible values of the control torque **M** belong to a ball [4]. This assumption does not contradict the mass distribution and the rigid body shape. We assume that

$$\mathbf{M} = b\mathbf{u}, \quad |\mathbf{u}| \le 1, \quad b = b(t, \mathbf{L}), \quad 0 < b_* \le b < b^* < \infty, \tag{1.2}$$

where b is a scalar function bounded in the considered domain of arguments t and $\mathbf{L}$ according to (1.2). This domain is either defined a priori or estimated by the initial data. The retarding resisting torque is a perturbation linear with respect to angular velocity. The mathematical model of controlled rotations of the rigid body is written in the form of the Euler equations (1.1). We pose the following minimum-time problem on rotation deceleration:

$$\boldsymbol{\omega}(t_0) = \boldsymbol{\omega}^0, \quad \boldsymbol{\omega}(T) = 0, \quad T - t_0 \to \min_{\mathbf{u}}, \quad |\mathbf{u}| \le 1. \tag{1.3}$$

We find the exact solution of problem (1.1)–(1.3) not assuming smallness of various parameters. It is required to synthesize an optimal control law $u = u(t, \boldsymbol{\omega})$, construct the corresponding trajectory $\boldsymbol{\omega}(t, t_0, \boldsymbol{\omega}^0)$, and find time $T = T(t_0, \boldsymbol{\omega}^0)$ and the Bellman function $W = T(t, \boldsymbol{\omega}) - t$, i.e., the current value of the functional.

2. SOLUTION OF THE OPTIMAL DECELERATION PROBLEM

We will solve the optimal control synthesis problem in a simplified statement. Note that the moment of the linear resistance forces is external. Based on the dynamic programming and Schwartz inequality, the minimum-time control law has the following form [4]:

$$M_p = -b\frac{Ap}{G}, \quad M_q = -b\frac{Bq}{G}, \quad M_r = -b\frac{Cr}{G}, \quad b = b(t, G). \tag{2.1}$$

Here, for the sake of simplification, we assume that $b = b(t, G)$ and $0 < b_1 \le b < b_2 < \infty$. Let us multiply the first equation in (1.1) by Ap, the second equation by Bq, and the third equation by Cr and add them together. We obtain

$$\dot{G} = -b(t, G) - \lambda G, \quad G(t_0) = G^0, \quad G(T, t_0, G^0) = 0, \quad T = T(t_0, G^0), \quad W(t, G) = T(t, G) - t. \tag{2.2}$$

Assuming that $b = b(t)$, i.e., function $b(t)$ does not depend on the magnitude of G, we obtain the following solution and condition for T:

$$G(t) = G^0 e^{-\lambda(t - t_0)} - \int_{t_0}^{t} b(\tau) e^{-\lambda(t - \tau)} d\tau, \quad G^0 = e^{-\lambda t_0} \int_{t_0}^{T} b(\tau) e^{\lambda\tau} d\tau, \quad T = T(t_0, G^0). \tag{2.3}$$

A solution always exists, which results in construction of a solution of the minimum-time problem in the form of synthesis. Here, t is the current time in the deceleration process, and T is the minimum time. For $b =$ const, solutions of Eq. (2.2) for G and the boundary value problem (2.3) are simplified and written as

$$G(t) = \frac{1}{\lambda}[(G^0\lambda + b)\exp(-\lambda t) - b], \quad T = \frac{1}{\lambda}\ln\left(G^0\frac{\lambda}{b} + 1\right), \quad t_0 = 0. \tag{2.4}$$

Below, case (2.4) is analyzed in detail. Let us multiply the first equation in (1.1) by p, the second equation by q, and the third equation by r and add them together. As a result, we obtain the expression for the derivative of the kinetic energy H

$$\dot{H} = -\frac{2bH}{G} - 2\lambda H, \quad H = \frac{1}{2}(Ap^2 + Bq^2 + Cr^2). \tag{2.5}$$

Since function $G(t)$ is known, Eq. (2.5) can be completely integrated to give

$$H = H^0 {G^0}^{-2} \lambda^{-2}[(G^0\lambda + b)\exp(-\lambda t) - b]^2. \tag{2.6}$$

Let, for definiteness, $A > B > C$. First, consider motion under the condition $2HA \ge G^2 > 2HB$, which corresponds to the trajectories encircling the axis of the largest inertia moment A. Let us introduce function k,

$$k^2 = \frac{(B - C)(2HA - G^2)}{(A - B)(G^2 - 2HC)}, \quad 0 \le k^2 \le 1, \tag{2.7}$$

which is the module of the elliptic functions [5] and is uniquely related to the kinetic energy H and angular momentum G. The value $k = 0$ corresponds to the rotation about axis A, and $k = 1$, to motion along the separatrix. Using formulas (2.7), (1.1), (2.1), (2.4), and (2.6), we obtain the expression for the derivative of k^2

$$\frac{dk^2}{dt} = \frac{2(G^0\lambda + b)\exp(-\lambda t)}{\sigma[(G^0\lambda + b)\exp(-\lambda t) - b]}(\alpha + \beta k^2 + \gamma k^4), \quad k^2(0) = k^{0^2},$$
$$\sigma = \lambda^{-1}(A-B)(B-C)(A-C), \quad \alpha = A(B-C)^2(1 - 2AH^0G^{0^{-2}}),$$
$$\beta = (A-B)(B-C)(A+C-4ACH^0G^{0^{-2}}), \quad \gamma = C(A-B)^2(1-2CH^0G^{0^{-2}}). \tag{2.8}$$

Stationary points k^{*2} correspond to positive roots of the equation $\alpha + \beta z + \gamma z^2 = 0$, $z = k^2$. Separating variables and integrating Eq. (2.8), we find implicit dependence of k^2 on time t:

$$\frac{\beta + 2\gamma k^2 - \sqrt{-\Delta}}{\beta + 2\gamma k^2 + \sqrt{-\Delta}} = \exp(4\lambda t)[(G^0\lambda + b)\exp(-\lambda t) - b]^{4\lambda t}, \tag{2.9}$$

where $\Delta = -(A - B)^2(B - C)^2(A - C)^2$. Formula (2.9) relates variables k^2 and t. It is easily solvable in k^2.

3. CONSTRUCTION OF OPTIMAL CONTROLLED MOTION

Let us solve system (1.1) in a different way. System (1.1) can be written in the vector form as follows:

$$\dot{\mathbf{L}} + \boldsymbol{\omega} \times \mathbf{L} = \mathbf{M} - \lambda\mathbf{L}, \quad \mathbf{L} = J\boldsymbol{\omega}. \tag{3.1}$$

Here, $\mathbf{L}$ is the angular momentum vector, $\boldsymbol{\omega}$ is the angular velocity vector, and $J = \mathrm{diag}(A, B, C)$ is the body inertia tensor. Let us denote

$$L_x = Ap, \quad L_y = Bq, \quad L_z = Cr. \tag{3.2}$$

Then,

$$\dot{L}_x = A\dot{p}, \quad \dot{L}_y = B\dot{q}, \quad \dot{L}_z = C\dot{r}, \tag{3.3}$$

where L_x, L_y, and L_z are projections of vector $\mathbf{L}$ onto the axes of the body frame $Oxyz$. With regard to (3.2) and (3.3), system (3.1) can be written as follows:

$$\dot{\mathbf{L}} + [J^{-1}\mathbf{L} \times \mathbf{L}] = -b\frac{\mathbf{L}}{G} - \lambda\mathbf{L}. \tag{3.4}$$

Let us make substitution $\mathbf{L} = G\mathbf{l}$, where G is the magnitude of the angular momentum and $\mathbf{l}$ is the unit vector of vector $\mathbf{L}$. As a result, we have

$$L(0) = L^0, \quad L(T) = 0, \quad \dot{\mathbf{L}} = \dot{G}\mathbf{l} + G\dot{\mathbf{l}}. \tag{3.5}$$

Substitution of formula (3.5) into (3.4) with regard to equality $\dot{C} = -b - \lambda G$ yields

$$G^{-1}\dot{\mathbf{l}} + \lfloor J^{-1}\mathbf{l} \times \mathbf{l} \rfloor = 0. \tag{3.6}$$

Let us replace argument t with τ. From (3.6), it finally follows that

$$\mathbf{l}' + \lfloor J^{-1}\mathbf{l} \times \mathbf{l} \rfloor = 0, \quad d\tau = G(t)dt, \quad l(0) = L^0/G^0. \tag{3.7}$$

Let us introduce the notation

$$\mathbf{K} = J^{-1}\mathbf{l}. \tag{3.8}$$

Substituting (3.8) into (3.7), we obtain the system similar to the Euler equations for a free rigid body:

$$J\mathbf{K}' + [\mathbf{K} \times J\mathbf{K}] = 0. \tag{3.9}$$

It can be completely integrated [1–5]. To this end, let us take the inner product of Eq. (3.9) with **K**:

$$(\mathbf{K}, J\mathbf{K}') = 0. \tag{3.10}$$

Integrating this equation, we obtain the expression similar to that for the kinetic energy [5]

$$H_k = \frac{1}{2}(\mathbf{K}, J\mathbf{K}) = \text{const}. \tag{3.11}$$

Taking the inner product of equation (3.9) with **l**, we obtain

$$(J\mathbf{K}, J\mathbf{K}) = G_k^2 = 1, \tag{3.12}$$

where G_k is the magnitude of the angular momentum **l**.

From (3.11) and (3.12), we express k_x^2 and k_z^2 in terms of k_y^2, A, B, C, H_k, and G_k:

$$\begin{aligned} k_x^2 &= \frac{1}{A(C-A)}[(2H_kC - G_k^2) - B(C-B)k_y^2], \\ k_z^2 &= \frac{1}{C(C-A)}[(G_k^2 - 2H_kA) - B(B-A)k_y^2]. \end{aligned} \tag{3.13}$$

Values k_x and k_z defined by (3.13) are substituted into the second equation in (3.9), which yields the following differential equation in k_y with separating variables:

$$\frac{dk_y}{d\tau} = \pm\frac{1}{B\sqrt{AC}}[(2H_kC - G_k^2) - B(C-B)k_y^2]^{1/2}[(G_k^2 - 2H_kA) - B(B-A)k_y^2]^{1/2}. \tag{3.14}$$

If Eq. (3.10) has been integrated, then functions k_x and k_z are found from (3.13). The sign of the radicals—plus or minus—is selected by means of Eqs. (3.9).

Let us consider two cases corresponding to different relations between constants H_k and G_k. For definiteness, we assume that $A > B > C$.

Consider the case where $2H_kA \geq G_k^2 > 2H_kB$. In this case, k_x is not equal to zero in the course of motion. To integrate Eqs. (3.14), we change variables

$$k_y = \pm\sqrt{\frac{2H_kA - G_k^2}{B(A-B)}}\sin\varsigma, \quad \tau' = \sqrt{\frac{(A-B)(G_k^2 - 2H_kC)}{ABC}}t \tag{3.15}$$

and introduce a positive parameter $0 \leq k^2 < 1$ by the formula

$$k^2 = \frac{(B-C)(2H_kA - G_k^2)}{(A-B)(G_k^2 - 2H_kC)}.$$

In the new variables, Eq. (3.14) is written as

$$\frac{d\varsigma}{d\tau'} = \sqrt{1 - k^2\sin^2\varsigma}.$$

Let $k_y = 0$ when $t = 0$. Then, $\varsigma = \mathrm{am}\tau'$, where am is the elliptic amplitude modulo k. Solutions of the Euler equations (3.9) in the considered case is written in terms of the Jacoby functions dn, sn, and cn as

$$k_x = \sqrt{\frac{G_k^2 - 2H_kC}{A(A-C)}}\mathrm{dn}(\tau'; k), \quad k_y = \pm\sqrt{\frac{2H_kA - G_k^2}{B(A-B)}}\mathrm{sn}(\tau'; k), \quad k_z = \mp\sqrt{\frac{2H_kA - G_k^2}{C(A-C)}}\mathrm{cn}(\tau'; k). \tag{3.16}$$

Taking into account that $\mathbf{l} = \mathbf{L}/G$ and using formulas (3.8) and (3.2), we obtain

$$p = G_k\sqrt{\frac{G_k^2 - 2H_kC}{A(A-C)}}\mathrm{dn}(\tau'; k), \quad q = \pm G_k\sqrt{\frac{2H_kA - G_k^2}{B(A-B)}}\mathrm{sn}(\tau'; k),$$
$$r = \mp G_k\sqrt{\frac{2H_kA - G_k^2}{C(A-C)}}\mathrm{cn}(\tau'; k). \tag{3.17}$$

Let us turn to spherical angles θ and φ characterizing projections of the angular momentum vector $\mathbf{L}$ onto the axes of the body coordinate system. Projections of vector $\boldsymbol{\omega}$ onto axes Ox, Oy, and Oz are given by

$$p = \frac{G_k}{A}\sin\theta\sin\varphi, \quad q = \frac{G_k}{B}\sin\theta\cos\varphi, \quad r = \frac{G_k}{C}\cos\theta. \tag{3.18}$$

Then, substituting (3.18) into (3.17), we obtain

$$\sin\theta\sin\varphi = \sqrt{\frac{A(G_k^2 - 2H_kC)}{(A-C)}}\mathrm{dn}(\tau'; k), \quad \sin\theta\cos\varphi = \pm\sqrt{\frac{B(2H_kA - G_k^2)}{(A-B)}}\mathrm{sn}(\tau'; k),$$
$$\cos\theta = \mp\sqrt{\frac{C(2H_kA - G_k^2)}{(A-C)}}\mathrm{cn}(\tau'; k). \tag{3.19}$$

Now, let us consider the case where $2H_kB > G_k^2 \geq 2H_kC$. In this case, k_z is not equal to zero in the course of motion. Let us change variables

$$k_y = \pm\sqrt{\frac{G_k^2 - 2H_kC}{B(B-C)}}\sin\varsigma, \quad \tau' = \sqrt{\frac{(B-C)(2H_kA - G_k^2)}{ABC}}t.$$

If we introduce parameter $0 \leq k^2 < 1$ by the formula

$$k^2 = \frac{(A-B)(G_k^2 - 2H_kC)}{(B-C)(2H_kA - G_k^2)},$$

then Eq. (3.14) takes the form

$$\frac{d\varsigma}{d\tau'} = \sqrt{1 - k^2\sin\varsigma}.$$

Let $k_y = 0$ when $t = 0$. Then, solutions of Eqs. (3.9) are given by

$$k_x = \mp\sqrt{\frac{G_k^2 - 2H_kC}{A(A-C)}}\mathrm{cn}(\tau'; k), \quad k_y = \pm\sqrt{\frac{G_k^2 - 2H_kC}{B(B-C)}}\mathrm{sn}(\tau'; k), \quad k_z = \sqrt{\frac{2H_kA - G_k^2}{C(A-C)}}\mathrm{dn}(\tau'; k). \tag{3.20}$$

Taking into account that $\mathbf{l} = \mathbf{L}/G$ and using formulas (3.8), (3.2), and (3.17), we obtain

$$\sin\theta\sin\varphi = \mp\sqrt{\frac{A(G_k^2 - 2H_kC)}{(A-C)}}\mathrm{cn}(\tau'; k), \quad \sin\theta\cos\varphi = \pm\sqrt{\frac{B(G_k^2 - 2H_kC)}{(B-C)}}\mathrm{sn}(\tau'; k).$$
$$\cos\theta = \sqrt{\frac{C(2H_kA - G_k^2)}{(A-C)}}\mathrm{dn}(\tau'; k). \tag{3.21}$$

Here, signs of the radicals are to be taken either both upper or both lower simultaneously. Note that, in both considered cases, k_x, k_y, and k_z are periodic functions of time; therefore, polhodes are closed curves.

CONCLUSIONS

A minimum-time control problem on deceleration of rotations of a dynamically asymmetric body in a resistant medium has been studied. An optimal control and minimum time (Bellman function) have been determined. The controlled motion is motion of the Euler–Poisson type with the body angular momentum G_k varying in time in accordance with formulas (2.3) and (2.4). Note that the approach discussed in this paper was developed in [4] on the basis of the theory of controlled systems with invariant norm [6–8].

REFERENCES

1. L. D. Akulenko, D. D. Leshchenko, and F. L. Chernous'ko, "Fast Rotation about a Fixed Point of a Heavy Solid Body in a Resisting Media," Izv. Ross. Akad. Nauk, Mekh. Tverd. Tela, No. 3 (1982).
2. V. N. Koshlyakov, *Problems of Solid Body Dynamics and Applied Gyroscope Theory. Analytical Methods* (Nauka, Moscow, 1985) [in Russian].
3. E. D. Raus, *Dynamics of a System of Solid Bodies* (Nauka, Moscow, 1983), Vol. 2 [in Russian].
4. L. D. Akulenko, *Asymptotic Methods of Optimal Control* (Nauka, Moscow, 1987) [in Russian].
5. A. P. Markeev, *Theoretical Mechanics* (Nauka, Moscow, 1990) [in Russian].
6. M. Athans and P. L. Falb, *Optimal Control: An Introduction to the Theory and Its Applications* (McGraw-Hill, 1966; Energiya, Moscow, 1968).
7. E. B. Lee and L. Marcus, *Foundations of Optimal Control Theory* (Wiley, New York, 1967; Nauka, Moscow, 1972).
8. A. M. Letov, *Flight Dynamics and Control* (Nauka, Moscow, 1969) [in Russian].

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