International Applied Mechanics, Vol. 35, No. 1, 1999

ON THE EVOLUTION OF RIGID-BODY ROTATIONS

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The perturbed rotational motion of a rigid body with a nearly Lagrangian mass distribution is studied. It is assumed that the angular velocity of the body is sufficiently high, its direction is close to the axis of dynamic symmetry of the body, and the perturbing moments are small in comparison with the gravity moment. A small parameter is introduced in a special manner and the acceleration method is used. Averaged systems of motion equations are obtained in first and second approximations. The evolution of the precession angle is determined in the second approximation.

1. Statement of Problem. We shall consider the motion of an asymmetric heavy rigid body about a fixed point O under the influence of gravity G alone. The equations of motion have the form [3]

$$a p^{\cdot} + (C - B) q r = m g (Z_c \sin\theta \cos\varphi - y_c \cos\theta);$$

$$B q^{\cdot} + (A - C) p r = m g (x_c \cos\theta - Z_c \sin\theta \sin\varphi);$$

$$C r^{\cdot} + (B - A) q p = m q (y_c \sin\theta \sin\varphi - x_c \sin\theta \cos\varphi); \tag{1.1}$$

$$\psi^{\cdot} = (p \sin\varphi + q \cos\varphi) \operatorname{cosec}\theta, \quad \theta^{\cdot} = p \cos\varphi - q \sin\varphi;$$

$$\varphi^{\cdot} = r - (p \sin\varphi + q \cos\varphi) \cot\theta .$$

We shall examine the case of a heavy rigid body in which the ellipsoid of inertia with respect to point O is close to the ellipsoid of rotation, so that its principal moments of inertia have the form

$$A = A^0 (1 + \varepsilon \delta_1), \quad B = A^0 (1 + \varepsilon \delta_2), \quad C \neq A^0 . \tag{1.2}$$

Here, δ_1 and δ_2 are dimensionless constants on the order of unity, A^0 is the characteristic value of the moments of inertia, and $\varepsilon << 1$ is a small parameter.

We assume that the coordinates of the center of gravity C with respect to the fixed point satisfy the relation

$$0 < (x_c^2 + y_c^2)^{1/2} << z_c . \tag{1.3}$$

Thus, dynamic symmetry is lost when the center of gravity of the body is displaced from the O_c axis, and for the coordinates of the center of gravity in the case in question we can write

$$x_c = \varepsilon x_1 l, \quad y_c = \varepsilon y_1 L, \quad Z_c = l, \tag{1.4}$$

where x_1 and y_1 are dimensionless quantities that are considered finite in comparison with the small parameter ε, and l is the characteristic dimension of the body.

Odessa Academy of Cold, Ukraine. Translated from Priknadnaya Mekhanika, Vol. 35, No. 1, pp. 98–103, January, 1999. Original article submitted July 15, 1997.

1063-7095/99/3501-0093$22.00

We reduce system of equations (1.1) to a system of Euler dynamic equations (1.5), relating to perturbations the gyroscopic moments and the moments due to displacement of the center of gravity of the body from the axis of dynamic symmetry [7]

$$A p^{\cdot} + (C - A) q r = k \sin\theta \cos\varphi + M_1 ; \tag{1.5}$$

$$A q^{\cdot} + (A - C) p r = -k \sin\theta \cos\varphi + M_2 ;$$

$$C r^{\cdot} = M_3 , \quad M_i = M_i (p, q, r, \psi, \theta, \varphi, t) \; (i = 1, 2, 3) .$$

The last three kinematic equations of (1.1) are unchanged. Here, M_i (i = 1, 2, 3) are projections of the vector of the perturbing moment onto the principal inertial axes that pass through point O; $k = mgl$.

Here, just as in [5], the following assumptions are made:

$$p^2 + q^2 << r^2 , \quad C r^2 >> k , \quad |M_i| << k \; (i = 1, 2, 3) . \tag{1.6}$$

Assumptions (1.6) mean that the direction of angular velocity of the body is close to the axis of dynamic symmetry; the angular velocity is high enough that the kinetic energy of the body is much greater than the potential energy due to the gravity moment; and the perturbing moments are small in comparison with the gravity moment. Inequalities (1.6) make it possible to introduce the small parameter and let

$$p = \varepsilon P , \; q = \varepsilon Q , \; k = \varepsilon K , \; \varepsilon << 1 ;$$

$$M_i = \varepsilon^2 M_i^* (P, Q, r, \psi, \theta, \varphi, t) \quad (i = 1, 2, 3). \tag{1.7}$$

A number of authors, such as [1, 2, 5, 6], have studied the nearly Lagrangian perturbed motion of a rigid body. The totality of simplifying assumptions (1.6) or (1.7), as was shown in [5], make it possible to obtain a comparatively simple averaging scheme in the general case.

The problem is to investigate the asymptotic behavior of system (1.5) if conditions (1.2), (1.4), (1.6), and (1.7) are satisfied. We shall employ the averaging method of [4] on a time interval on the order of ε^{-1}.

2. Averaging Procedure. We replace the variables and parameters (1.2), (1.4), and (1.7) in system (1.1). Having cancelled ε on both sides of the first two equations of (1.1), after a number of transformations we obtain a system in which A is replaced by A^0 in the first two equations, and the projections of the vector of the perturbing moment on the principal inertial axes passing through point O, following (1.5)–(1.7), have the form

$$M_1^* = -K \delta_1 \sin\theta \cos\varphi - K y_1 \cos\theta + Q r [\delta_1 (C - A^0) + \delta_2 A^0] ;$$

$$M_2^* = -K \delta_2 \sin\theta \sin\varphi + K x_1 \cos\theta + P r [\delta_2 (A^0 - C) - \delta_1 A^0] ; \tag{2.1}$$

$$M_3^* = K \sin\theta (y_1 \sin\varphi - x_1 \cos\varphi) .$$

An averaging procedure for a system such as (1.5) is described in [5]. We examine the zeroth-approximation system (having cancelled ε on both sides of the first two equations of (1.1) after replacement of the variables and parameters (1.2), (1.4), and (1.7)) and let ε = 0.

Then, from the last four equations obtained we have

$$r = r_0 , \; \psi = \psi_0 , \; \theta = \theta_0 , \; \varphi = r_0 t + \varphi_0 . \tag{2.2}$$

Here, r_0, ψ_0, θ_0, and φ_0 are constants equal to the initial values of the corresponding variables for $t = 0$. We substitute equalities (2.2) into the first two equations of system (1.1) with allowance for (1.2), (1.4), and (1.7) at ε = 0 and integrate the obtained system of two linear equations for P and Q. We represent the solution as

$$P = a\cos\gamma_0 + b\sin\gamma_0 + KC^{-1}r_0^{-1}\sin\theta_0\sin(r_0 t + \varphi_0);\quad (2.3)$$

$$Q = a\sin\gamma_0 - b\cos\gamma_0 + KC^{-1}r_0^{-1}\sin\theta_0\cos(r_0 t + \varphi_0);$$

$$a = P_0 - KC^{-1}r_0^{-1}\sin\theta_0\sin\varphi_0, \quad b = -Q_0 + KC^{-1}r_0^{-1}\sin\theta_0\cos\varphi_0;$$

$$\gamma_0 = n_0 t, \quad n_0 = (C - A^0)(A^0)^{-1}r_0 \neq 0, \quad |n_0/r_0| \leq 1.$$

Here, P_0 and Q_0 are the initial values of the new variables P and Q introduced in accordance with (1.7), and the variable $\gamma = \gamma_0$ has the meaning of the oscillation phase. System (1.1) with allowance for (1.2), (1.4), and (1.7) is substantially nonlinear, therefore, we introduce the additional variable γ, which is defined as

$$\gamma^{\cdot} = n, \quad \gamma(0) = 0, \quad n = (C - A^0)(A^0)^{-1}r. \quad (2.4)$$

Equalities (2.2) and (2.3) determine the general solution of system (1.1) with allowance for (1.2), (1.4), (1.7), and (2.4) for $\varepsilon = 0$. The first two relations of (2.3) can, with allowance for (2.2), be rewritten in equivalent form

$$P = a\cos\gamma + b\sin\gamma + KC^{-1}r^{-1}\sin\theta\sin\varphi;\quad (2.5)$$

$$Q = a\sin\gamma - b\cos\gamma + KC^{-1}r^{-1}\sin\theta\cos\varphi.$$

These equalities are easily solved for a and b.

We introduce a new variable ρ as follows:

$$r = r_0 + \varepsilon\rho. \quad (2.6)$$

Using formulas (2.5) and (2.6), we move in system (1.1) with allowance for (1.2), (1.4), (1.7), and (2.4) for $\varepsilon \neq 0$ fromvariables$P$, Q, r, ψ, θ, φ, and γ to the new variables a, b, ρ, ψ, θ, α, and γ, where

$$\alpha = \gamma + \varphi. \quad (2.7)$$

After transformations, we have a system of seven equations that is more convenient for further study:

$$a^{\cdot} = \varepsilon(A^0)^{-1}(M_1^0\cos\gamma + M_2^0\sin\gamma) - \varepsilon KC^{-1}r_0^{-1}\cos\theta\,(b - KC^{-1}r_0^{-1}\sin\theta\cos\alpha) +$$

$$+ \varepsilon^2 KC^{-1}r_0^{-2}\rho\cos\theta\,(b - 2KC^{-1}r_0^{-1}\sin\theta\cos\alpha) + \varepsilon^2 KC^{-2}r_0^{-2}M_3^0\sin\theta\sin\alpha;\quad (2.8)$$

$$b^{\cdot} = \varepsilon(A^0)^{-1}(M_1^0\sin\gamma - M_2^0\cos\gamma) + \varepsilon KC^{-1}r_0^{-1}\cos\theta\,(a + KC^{-1}r_0^{-1}\sin\theta\sin\alpha) -$$

$$- \varepsilon^2 KC^{-1}r_0^{-2}\rho\cos\theta\,(a + 2KC^{-1}r_0^{-1}\sin\theta\sin\alpha) - \varepsilon^2 KC^{-2}r_0^{-2}M_3^0\sin\theta\cos\alpha;$$

$$\rho^{\cdot} = \varepsilon C^{-1}M_3^0, \quad \theta^{\cdot} = \varepsilon(a\cos\alpha + b\sin\alpha);$$

$$\psi^{\cdot} = \varepsilon\,\mathrm{cosec}\,\theta\,(a\sin\alpha - b\cos\alpha) + \varepsilon KC^{-1}r_0^{-1} - \varepsilon^2 KC^{-1}r_0^{-2}\rho;$$

$$\alpha^{\cdot} = C(A^0)^{-1}r_0 + \varepsilon C(A^0)^{-1}\rho - \varepsilon\cot\theta\,(a\sin\alpha - b\cos\alpha) - \varepsilon KC^{-1}r_0^{-1}\cos\theta + \varepsilon^2 KC^{-1}r_0^{-2}\rho\cos\theta;$$

$$\gamma^{\cdot} = n_0 + \varepsilon(C - A^0)(A^0)^{-1}\rho.$$

Here, M_i^0 are functions obtained from M_i^* (see (1.7)) as a result of substitutions (2.5)–(2.7)

$$M_i^0(a,b,\rho,\psi,\theta,\alpha,\gamma,t)=M_i^*(P,Q,r,\psi,\theta,\varphi,t) \quad (i=1,2,3). \tag{2.9}$$

System (2.8) can be reduced to the form

$$\begin{aligned} &x^{\cdot}=\varepsilon F_1(x,y)+\varepsilon^2 F_2(x,y), \quad x(0)=x_0; \\ &y^{1\cdot}=\omega_1+\varepsilon g_1(x,y)+\varepsilon^2 g_2(x,y), \quad y^1(0)=y^{10}; \\ &y^{2\cdot}=\omega_2+\varepsilon h_1(x,y)+\varepsilon^2 h_2(x,y), \quad y^2(0)=y^{20}, \end{aligned} \tag{2.10}$$

where the vector function $x=(x^1,\dots,x^5)$ is composed of the slow variables a, b, ρ, ψ, and θ; y^1 and y^2 are the fast variables α and γ; and ω_1 and ω_2 are constant phases equal to $C(A^0)^{-1}r_0$ and $(C-A^0)(A^0)^{-1}r_0$, respectively. The vector functions F_i, g_i, and h_i ($i=1,2$) are determined by the right sides of Eqs. (2.8).

We let Z_1 be the two-dimensional vector (g_1, h_1). Since the perturbing moments M_i^* ($i=1,2,3$) are periodic with respect to φ with a period of 2π, then, according to substitutions (2.5)–(2.7), functions M_i^0 of (2.9) will be periodic functions of α and γ with periods of 2π.

In accordance with the procedure for constructing an asymptotic form of system (2.10) [4], we seek a substitution of variables

$$\begin{aligned} &x=x^*+\varepsilon u_1(x^*,y^*)+\varepsilon^2 u_2(x^*,y^*)+\dots; \\ &y=y^*+\varepsilon v_1(x^*,y^*)+\varepsilon^2 v_2(x^*,y^*)+\dots; \\ &y=(y^1,y^2),\ x^*=(x^{*1},\dots,x^{*5}),\ y^*=(y^{*1},y^{*2}) \end{aligned}$$

such that system (2.10) in the new variables takes the form

$$\begin{aligned} &x^{*\cdot}=\varepsilon A_1(x^*)+\varepsilon^2 A_2(x^*)+\dots; \\ &y^{*\cdot}=\omega+\varepsilon B_1(x^*)+\varepsilon B_2(x^*)+\dots, \quad \omega=(\omega_1,\omega_2). \end{aligned} \tag{2.11}$$

It is known [4] that the equations for the vector functions u_1 and v_1 have the form

$$\begin{aligned} &\omega\, \partial u_1/\partial y^*=F_1(x^*,y^*)-A_1(x^*); \\ &\omega\, \partial v_1/\partial y^*=Z_1(x^*,y^*)-B_1(x^*), \end{aligned} \tag{2.12}$$

where $(\partial f/\partial x)$ is the matrix of partial derivatives $\|\partial f_i/\partial x^j\|$ $(i,j=1,\dots,5)$. The functions $A_1(x^*)$ and $B_1(x^*)$ are determined by the formulas

$$\begin{aligned} &A_1(x^*)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} F_1(x^*,y^*)\,dy^{*1}\,dy^{*2}; \\ &B_1(x^*)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} Z_1(x^*,y^*)\,dy^{*1}\,dy^{*2}. \end{aligned} \tag{2.13}$$

The function $u_2(x^*,y^*)$ must be a solution of the equation

$$\partial u_2/\partial y^* \omega = G(x^*, y^*) - A_2(x^*);$$

$$G(x^*, y^*) = F_2(x^*, y^*) + \partial F_1/\partial^* u_1 + \partial F_1/\partial y^* v_1 - \partial u_1/\partial x^* A_1(x^*) - \partial u_1/\partial y^* B_1(x^*). \quad (2.14)$$

The function $A_2(x^*)$ is determined by the formula

$$A_2(x^*) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} G(x^*, y^*)\, dy^{*1}\, dy^{*2}. \quad (2.15)$$

We determine the averaged first-approximation system of equations for the slow variables

$$x_1^{*\cdot} = \varepsilon A_1(x_1^*), \quad x_1^*(0) = x_{10}, \quad (2.16)$$

the second-approximation system for the slow variables

$$x_2^{*\cdot} = \varepsilon A_1(x_2^*) + \varepsilon^2 A_2(x_2^*), \quad x_2^*(0) = x_{20}, \quad (2.17)$$

and the second-approximation system for the fast variables

$$y_2^{*\cdot} = \omega + \varepsilon B_1(x_1^*(t)), \quad y_2^*(0) = y^0, \quad y^0 = (y^{10}, y^{20}), \quad (2.18)$$

which is immediately integrated

$$y_2^*(t) = y^0 + \omega t + \varepsilon \int_0^t B_1(x_1^*(s))\, ds. \quad (2.19)$$

We determine the vector functions

$$x_\varepsilon^{\text{v}}(t) = x^{(1)}(\varepsilon t) + \varepsilon x^{(2)}(\varepsilon t) + \varepsilon u_1(x^{(1)}(\varepsilon t), y^0 + \omega t + \varepsilon \int_0^t B_1(x^{(1)}(\varepsilon s))\, ds); \quad (2.20)$$

$$y_\varepsilon^{\text{v}}(t) = y^0 + \omega t + \varepsilon \int_0^t B_1(x^{(1)}(\varepsilon s))\, ds.$$

Thus, the construction of approximate solutions $x_\varepsilon^{\text{v}}(t)$ and $y_\varepsilon^{\text{v}}(t)$ comes down to the following procedure: we solve Eqs. (2.12) and (2.14) by means of Fourier series, use formula (2.15) to construct the vector function $A_2(x^*)$, then determine solutions $x^{(1)}$ and $x^{(2)}$ in accordance with [5], and, finally, we obtain the desired approximations from formula (2.20).

3. The Case of a Nearly Dynamically Symmetrical Body. As an example of application of the proposed method, we study the motion of a heavy rigid body about a fixed point when relations (1.2) are satisfied for the principal moments of inertia and (1.4) for the coordinates of the center of gravity C with respect to the fixed point.

In this case, the first three equations of (2.8) in variables $a, b, \rho, \psi, \theta, \alpha, \gamma$ are written with allowance for (2.1); the other equations of system (2.8) remain unchanged. After calculations by formulas (2.13), the components of vector functions A_1 and B_1 have the form

$$A_1^{(1)} = -b\,[1/2\, C (A^0)^{-1} r_0 (\delta_1 + \delta_2) + K C^{-1} r_0^{-1} \cos\theta];$$

$$A_1^{(2)} = a\,[1/2\, C (A^0)^{-1} r_0 (\delta_1 + \delta_2) + K C^{-1} r_0^{-1} \cos\theta];$$

$$A_1^{(3)}=0,\quad A_1^{(4)}=KC^{-1}r_0^{-1},\quad A_1^{(5)}=0; \tag{3.2}$$

$$B_1^{(1)}=C(A^0)^{-1}\rho-KC^{-1}r_0^{-1}\cos\theta,\quad B_1^{(2)}=(C-A^0)(A^0)^{-1}\rho.$$

The fourth and fifth components of the vector function $u_1=\{u_1^{(i)}\}$ $(i=1,\ldots,5)$ are expressed as follows:

$$u_1^{(4)}=-C^{-1}A^0r_0^{-1}\operatorname{cosec}\theta\,(a\cos\alpha+b\sin\alpha);$$

$$u_1^{(5)}=C^{-1}A^0r_0^{-1}(a\sin\alpha-b\cos\alpha). \tag{3.3}$$

We determine the function $A_2(x^*)$ by formula (2.15)

$$A_2^{(1)}=KC^{-1}r_0^{-2}b\cos\theta\,[\rho-1/2\,KC^{-2}r_0^{-2}A^0(1+\cos\theta)]+1/4\,KC^{-1}r_0^{-1}b(\delta_1+\delta_2)\cos\theta;$$

$$A_2^{(2)}=-KC^{-1}r_0^{-2}a\cos\theta\,[\rho-1/2\,KC^{-2}r_0^{-2}A^0(1+\cos\theta)]-1/4\,KC^{-1}r_0^{-1}a(\delta_1+\delta_2)\cos\theta; \tag{3.4}$$

$$A_2^{(3)}=0,\ A_2^{(4)}=-KC^{-1}r_0^{-2}\rho+A^0K^2C^{-3}r_0^{-3}\cos\theta-1/2\,KC^{-1}r_0^{-1}(\delta_1+\delta_2),\quad A_2^{(5)}=0.$$

We find a solution of the averaged first-approximation system of equations (2.16) with allowance for (3.2) for the slow and fast variables:

$$a^{(1)}=a^0\cos\eta t-b^0\sin\eta t,\quad b^{(1)}=b^0\cos\eta t+a^0\sin\eta t;$$

$$\rho^{(1)}=0,\quad \psi^{(1)}=\varepsilon KC^{-1}r_0^{-1}t+\psi_0,\quad \theta^{(1)}=\theta_0; \tag{3.5}$$

$$\alpha^{(1)}=C(A^0)^{-1}r_0t-\varepsilon tKC^{-1}r_0^{-1}\cos\theta_0+\varphi_0,\quad \gamma^{(1)}=n_0t,$$

where

$$\eta=\varepsilon\,[1/2\,C(A^0)^{-1}r_0(\delta_1+\delta_2)+KC^{-1}r_0^{-1}\cos\theta_0];$$

a^0, b^0, n_0 are determined according to formula (2.3).

On the basis of the formulas, we can, following (2.20), construct the components of the function $x_\varepsilon^{\vee}(t)$ that satisfy variables ψ and θ

$$\psi_\varepsilon^{\vee}(t)=\psi_0+\varepsilon KC^{-1}r_0^{-1}t+V^{(1)};$$

$$V^{(1)}=\varepsilon^2tA^0K^2C^{-3}r_0^{-3}-1/2\,\varepsilon^2tKC^{-1}r_0^{-1}(\delta_1+\delta_2)-$$

$$-\varepsilon C^{-1}A^0r_0^{-1}\operatorname{cosec}\theta_0\,(a^{02}+b^{02})^{1/2}\sin(\alpha^{(1)}+\beta); \tag{3.6}$$

$$\sin\beta=a^{(1)}(a^{02}+b^{02})^{-1/2},\quad \theta_\varepsilon^{\vee}(t)=\theta_0.$$

Here, in the expression for $\psi_\varepsilon^{\vee}$, the bounded oscillating term contains nonzero initial data. The nature of the slow phase variation of small oscillations is evident from formulas (3.5) for $a^{(1)}$ and $\alpha^{(1)}$.

The obtained expression for $V^{(1)}$ refines for the given problem the formula for the angular velocity of precession $\omega_p=KC^{-1}r_0^{-1}$, which is found in the approximate theory of gyroscopes [3].

Note that in formula (3.6) for $V^{(1)}$, there is no dependence on the deviation of the center of gravity from the Oz_c axis, which is specified by expressions (1.14). The dimensionless quantities x_1 and y_1 disappear with averaging. In addition, these perturbations do not change the nutation angle, even in the second approximation.

REFERENCES

1. L. D. Akulenko, D. D. Leshchenko, and F. L. Chernous'ko, "Perturbed motion of a rigid body that is close to the Lagrange case," *Prikl. Mat. Mekh.*, **43**, No. 5, 771-778 (1979).
2. L. D. Akulenko, D. D. Leshchenko, and F. L. Chernous'ko, "Perturbed motion of a rigid body that is close to regular precession," *Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela*, No.5, 3-10 (1986).
3. N. N. Bukhgol'ts, *A Basic Course in Theoretical Mechanics* [in Russian], Part 2, Nauka, Moscow (1969).
4. V. M. Volosov and B. I. Morgunov, *Averaging Method in the Theory of Nonlinear Oscillatory Systems* [in Russian], Izd. MGU, Moscow (1971).
5. D. D. Leshchenko and A. S. Shamaev, "Perturbed rotational motion of a rigid body that is close to regular precession in the Lagrange case," *Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela*, No. 6, 8-17 (1987).
6. V. V. Sazonov and V. V. Sidorenko, "Perturbed motion of a rigid body that is close to Lagrange regular precession," *Prikl. Mat. Mekh.*, **54**, No. 6, 951-957 (1990).
7. F. L. Chernous'ko, L. D. Akulenko, and B. N. Sokolov, *Vibration Control* [in Russian], Nauka, Moscow (1980).