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PERTURBED MOTIONS OF A RIGID BODY THAT ARE CLOSE TO REGULAR PRECESSION

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The authors investigate perturbed rotational motions of a rigid body that are close to regular precession in the Lagrange case. It is assumed that the angular velocity of the body is large; its direction is close to the axis of dynamic symmetry of the body, and that two projections of the vector of the perturbing moment onto the principal axes of inertia of the body are small as compared to the restoring moment, while the third is of the same order of magnitude as the moment in question. A small parameter is introduced in a special way; the averaging method is employed. The averaged system of equations of motion is obtained in first approximation. Examples are considered.

1. Consider the motion of a dynamically symmetrical rigid body about fixed point O under the action of restoring and perturbing moments. The equations of motion (dynamic and kinematic Euler equations) have the form

$$\begin{gathered} Ap^{\cdot}+(C-A)qr=k\sin\theta\cos\varphi+M_1 \\ Aq^{\cdot}+(A-C)pr=-k\sin\theta\sin\varphi+M_2 \\ Cr^{\cdot}=M_3,\quad M_i=M_i(p,q,r,\psi,\theta,\varphi,t),\ (i=1,2,3) \\ \psi^{\cdot}=(p\sin\varphi+q\cos\varphi)\operatorname{cosec}\theta,\quad \theta^{\cdot}=p\cos\varphi-q\sin\varphi \\ \varphi^{\cdot}=r-(p\sin\varphi+q\cos\varphi)\operatorname{ctg}\theta \end{gathered} \tag{1.1}$$

Dynamic equations (1.1) are written in projections onto the principal axes of inertia of the body, passing through point O. Here p, q, r are the projections of the angular velocity vector of the body onto these axes; M_i (i = 1, 2, 3) are the projections of the vector of the perturbing moment onto these same axes, which are 2π-periodic functions of the Euler angles ψ, θ, φ; ψ is the precession angle; θ is the nutation angle; φ is the angle of intrinsic rotation or spin; and A and C are the equatorial and axial moments of inertia of the body relative to point O, $A \neq C$. It is assumed that the body is acted upon by a restoring moment whose maximum value is equal to k and that is generated by a force of constant magnitude and direction, applied at some fixed point of the axis of dynamic symmetry. In the case of a heavy top we have

$$k=mgl \tag{1.2}$$

Here m is the mass of the body; g is the acceleration due to gravity; and l is the distance from fixed point O to the center of gravity of the body.

The perturbing moments M_i in (1.1) are assumed to be known functions of their arguments. For M_i = 0 (i = 1, 2, 3) Eqs. (1.1) correspond to the Lagrange case and may describe motions of a Lagrange top acted upon by perturbations of various physical origin, as well as motions of a free rigid body relative to the center of mass, when this body is acted upon by a restoring moment generated by aerodynamic forces, and certain perturbing moments.

In this paper we make the following initial assumptions:

$$p^2+q^2\ll r^2,\quad Cr^2\gg k,\quad |M_i|\ll k\ (i=1,2),\quad M_3\sim k \tag{1.3}$$

which mean that the direction of the angular velocity of the body is close to the axis of dynamic symmetry; the angular velocity is large, so that the kinetic energy of the body is much greater than the potential energy resulting from the restoring moment; two projections of the vector of the perturbing moment onto the principal axes of inertia of the body are small as compared to the restoring moment, while the third is of the same order of magnitude as this moment. Inequalities (1.3) allow us to introduce the small parameter ε and to set

$$p=\varepsilon P, \quad q=\varepsilon Q, \quad k=\varepsilon K, \quad \varepsilon \ll 1$$
$$M_i=\varepsilon^2 M_i^*(P, Q, r, \psi, \theta, \varphi, t) \quad (i=1, 2), \quad M_3=\varepsilon M_3^*(P, Q, r, \psi, \theta, \varphi, t) \tag{1.4}$$

The new variables P and Q, as well as the variables and constants r, ψ, θ, ϕ, K, A, C, M_i^* ($i = 1, 2, 3$), are assumed to be bounded quantities of order unity as $\varepsilon \to 0$.

The problem that we formulate is that of investigating the asymptotic behavior of the solutions of system (1.1) for small ε, if conditions (1.3) and (1.4) are satisfied. This will be done by employing the averaging method [1-3], which is extensively employed in problems of dynamics of rigid bodies, on a time interval of order ε^{-1}.

In [3-5], this method was employed to investigate a variety of problems of dynamics, chiefly for bodies with dynamic symmetry. Paper [6] was the first to perform averaging with respect to Euler-Poinsot motion for an asymmetrical body. A number of studies, e.g., [3, 5, 7-14], have investigated perturbed motions close to Lagrange motion. The ensemble of simplifying assumptions (1.3) or (1.4), made in this paper, enables us to obtain a relatively simple averaging scheme in the general case, and to exhaustively investigate a number of examples.

2. In system (1.1) we make change of variables (1.4); canceling ε on both sides of the first two equations in (1.1), we obtain

$$\begin{gathered}
AP^{\cdot}+(C-A)Qr=K\sin\theta\cos\varphi+\varepsilon M_1^* \\
AQ^{\cdot}+(A-C)Pr=-K\sin\theta\sin\varphi+\varepsilon M_2^*, \quad Cr^{\cdot}=\varepsilon M_3^* \\
\psi^{\cdot}=\varepsilon(P\sin\varphi+Q\cos\varphi)\operatorname{cosec}\theta, \quad \theta^{\cdot}=\varepsilon(P\cos\varphi-Q\sin\varphi) \\
\varphi^{\cdot}=r-\varepsilon(P\sin\varphi+Q\cos\varphi)\operatorname{ctg}\theta
\end{gathered} \tag{2.1}$$

Let us consider the zero-approximation system; we set $\varepsilon = 0$ in (2.1). Then the last four equations in (2.1) yield

$$r=r_0, \quad \psi=\psi_0, \quad \theta=\theta_0, \quad \varphi=r_0 t+\varphi_0 \tag{2.2}$$

Here r_0, ψ_0, θ_0, ϕ_0 are constants equal to the initial values of the corresponding variables for $t = 0$. We substitute (2.2) into the first two equations of system (2.1) for $\varepsilon = 0$, and we integrate the resultant system of two linear equations for P, Q. We write the solution in the form

$$\begin{gathered}
P=a\cos\gamma_0+b\sin\gamma_0+KC^{-1}r_0^{-1}\sin\theta_0\sin(r_0 t+\varphi_0) \\
Q=a\sin\gamma_0-b\cos\gamma_0+KC^{-1}r_0^{-1}\sin\theta_0\cos(r_0 t+\varphi_0) \\
a=P_0-KC^{-1}r_0^{-1}\sin\theta_0\sin\varphi_0, \quad b=-Q_0+KC^{-1}r_0^{-1}\sin\theta_0\cos\varphi_0 \\
\gamma_0=n_0 t, \quad n_0=(C-A)A^{-1}r_0\neq 0, \quad |n_0/r_0|\ll 1
\end{gathered} \tag{2.3}$$

Here P_0, Q_0 are the initial values of the new variables P, Q, introduced in accordance with (1.4), while the variable $\gamma = \gamma_0$ has the meaning of the oscillation phase. System (2.1) is essentially nonlinear (the natural oscillation frequency of the variables P, Q depends on the slow variable r), and therefore we introduce the additional variable γ, defined by the equation

$$\gamma^{\cdot}=n, \quad \gamma(0)=0 \tag{2.4}$$

For $\varepsilon = 0$ we have $\gamma = \gamma_0 = n_0 t$ in accordance with (2.3). Equations (2.2) and (2.3) define the general solution of system (2.1), (2.4) for $\varepsilon = 0$. By eliminating the constants, with allowance for (2.2), it is possible to rewrite the first two expressions in (2.3) in equivalent form:

$$P = a\cos\gamma + b\sin\gamma + KC^{-1}r^{-1}\sin\theta\sin\varphi$$
$$Q = a\sin\gamma - b\cos\gamma + KC^{-1}r^{-1}\sin\theta\cos\varphi \tag{2.5}$$

and to solve for a, b:

$$a = P\cos\gamma + Q\sin\gamma - KC^{-1}r^{-1}\sin\theta\sin(\gamma+\varphi)$$
$$b = P\sin\gamma - Q\cos\gamma + KC^{-1}r^{-1}\sin\theta\cos(\gamma+\varphi) \tag{2.6}$$

Let us consider system (2.1) for $\varepsilon \neq 0$ and expressions (2.5) and (2.6) as change-of-variable formulas (that contain the variable γ) that define a changeover from variables P, Q to variables a, b of Van der Pol type [1] and vice versa. Using these formulas, in system (2.1), (2.4) we convert from the variables P, Q, r, ψ, θ, ϕ, γ to the new variables a, b, r, ψ, θ, α, γ, where

$$\alpha = \gamma + \varphi \tag{2.7}$$

After some manipulation, we obtain a system of seven equations that is more convenient for subsequent investigation (instead of the six in (2.1)):

$$a^{\cdot} = \varepsilon A^{-1}(M_1^0\cos\gamma + M_2^0\sin\gamma) - \varepsilon KC^{-1}r^{-1}\cos\theta(b -$$
$$- KC^{-1}r^{-1}\sin\theta\cos\alpha) + \varepsilon KC^{-2}r^{-2}M_3^0\sin\theta\sin\alpha$$
$$b^{\cdot} = \varepsilon A^{-1}(M_1^0\sin\gamma - M_2^0\cos\gamma) + \varepsilon KC^{-1}r^{-1}\cos\theta(a +$$
$$+ KC^{-1}r^{-1}\sin\theta\sin\alpha) - \varepsilon KC^{-2}r^{-2}M_3^0\sin\theta\cos\alpha, \quad r^{\cdot} = \varepsilon C^{-1}M_3^0$$
$$\psi^{\cdot} = \varepsilon\,\mathrm{cosec}\,\theta(a\sin\alpha - b\cos\alpha) + \varepsilon KC^{-1}r^{-1}$$
$$\theta^{\cdot} = \varepsilon(a\cos\alpha + b\sin\alpha) \tag{2.8}$$
$$\alpha^{\cdot} = CA^{-1}r - \varepsilon\,\mathrm{ctg}\,\theta(a\sin\alpha - b\cos\alpha) - \varepsilon KC^{-1}r^{-1}\cos\theta, \quad \gamma^{\cdot} = (C-A)A^{-1}r$$

Here M_i^0 denotes functions obtained from M_i^* (see (1.4)) as a result of substitution (2.5)-(2.7), i.e.,

$$M_i^0(a, b, r, \psi, \theta, \alpha, \gamma, t) = M_i^*(P, Q, r, \psi, \theta, \varphi, t) \quad (i = 1, 2, 3) \tag{2.9}$$

Note that the changeover from two variables P, Q to three variables a, b, γ is due to reasons of convenience: for $\varepsilon = 0$ the system for P, Q has the form of a linear system, while substitution (2.5) is nonsingular for all a, b.

We introduce vector x, whose components are provided by the slow variables a, b, r, ψ, θ of system (2.8). Then this system can be written in the form

$$x^{\cdot} = \varepsilon X(x, \alpha, \gamma, t), \quad \alpha^{\cdot} = CA^{-1}r + \varepsilon Y(x, \alpha)$$
$$\gamma^{\cdot} = (C-A)A^{-1}r, \quad x(0) = x_0, \quad \alpha(0) = \alpha_0, \quad \gamma(0) = 0 \tag{2.10}$$

Here the vector-valued function X and scalar function Y are defined by the right sides of (2.8), whose initial values can be obtained in accordance with (2.2)-(2.4), (2.7).

Consider system (2.8) or (2.10) from the standpoint of employing the averaging method of [1-3]. System (2.8) contains the slow variables a, b, r, ψ, θ and fast variables represented by the phases α, γ and time t; γ appears only in the first three equations of (2.8). The system is essentially nonlinear, and it is extremely difficult to employ the averaging method directly [15]. Let us assume, for the sake of simplicity, that the perturbing moments M_i^* are independent of t. Since M_i^* ($i = 1, 2, 3$) are 2π-periodic in ϕ, it follows, in accordance with (2.5)-(2.7), that functions M_i^0 from (2.9) will be 2π-periodic functions of α and γ. Then system (2.10) contains two rotating phases α and γ and the corresponding frequencies $CA^{-1}r$ and $(C - A)A^{-1}r$ are variable. In averaging system (2.8) or (2.10), two cases should be distinguished: the nonresonant case, when frequencies $CA^{-1}r$ and $(C - A)\cdot A^{-1}r$ are noncomensurable, and the resonant case, when these frequencies are comensurable [15]. A very important feature of system (2.10) is the fact that the ratio of the frequencies is constant $[(C-A)A^{-1}r]/[CA^{-1}r]=1-AC^{-1}$ and the resonant case occurs for

$$C/A=i/j, \quad i/j\leq 2 \tag{2.11}$$

where i and j are relatively prime natural numbers, while in the nonresonant case C/A is an irrational number. As a result of (2.11), averaging of nonlinear system (2.10), in which X is independent of t, is equivalent to averaging of a quasi-linear system with constant frequencies. This can be achieved by introducing the independent variable γ.

In the nonresonant case ($C/A \neq i/j$) we obtain the first-approximation averaged system by independent averaging of the right sides of system (2.8) with respect to both fast variables α, γ. As a result, we obtain the following equations for the slow variables:

$$\begin{gathered}
a^{\cdot}=\varepsilon A^{-1}\mu_1-\varepsilon KC^{-1}r^{-1}b\cos\theta+\varepsilon KC^{-2}r^{-2}\sin\theta\mu_4^{\circ} \\
b^{\cdot}=\varepsilon A^{-1}\mu_2+\varepsilon KC^{-1}r^{-1}a\cos\theta-\varepsilon KC^{-2}r^{-2}\sin\theta\mu_5^{\circ} \\
r^{\cdot}=\varepsilon C^{-1}\mu_3,\quad \psi^{\cdot}=\varepsilon KC^{-1}r^{-1},\quad \theta^{\cdot}=0 \\
\mu_1(a,b,r,\psi,\theta)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}(M_1^{\circ}\cos\gamma+M_2^{\circ}\sin\gamma)\,d\alpha\,d\gamma \\
\mu_2(a,b,r,\psi,\theta)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}(M_1^{\circ}\sin\gamma-M_2^{\circ}\cos\gamma)\,d\alpha\,d\gamma, \\
\mu_3(a,b,r,\psi,\theta)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}M_3^{\circ}\,d\alpha\,d\gamma \\
\mu_4^{\circ}(a,b,r,\psi,\theta)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}M_3^{\circ}\sin\alpha\,d\alpha\,d\gamma, \\
\mu_5^{\circ}(a,b,r,\psi,\theta)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}M_3^{\circ}\cos\alpha\,d\alpha\,d\gamma
\end{gathered} \tag{2.12}$$

Solving averaged system (2.12) for perturbing moments of specific form, we can determine the motion of the body in the nonresonant case with an error of order ε on an interval of time variation of order ε^{-1}. Note that the last equation in system (2.12) can be integrated; it yields $\theta = \theta_0$.

The above system is equivalent to a two-frequency system with constant frequencies, since both frequencies are proportional to the axial component r of the angular velocity vector. Therefore the applicability of the averaging method can be substantiated in the same way as for a quasi-linear system. The principal assertion involves the following. Assume that function X is sufficiently smooth with respect to α, γ, and that it satisfies a Lipschitz condition with respect to x, with a constant that is independent of α, γ. Then on the plane of permissible values of the parameters C, A there exists a set L of measure zero such that if $(C, A)\notin L$, then for the solutions of systems (2.10) and (2.12) we have the bound $|x(t,\varepsilon)-\xi(\varepsilon t)|\leq D\varepsilon$, $t\in[0, \Theta\varepsilon^{-1}]$, in which $\xi(\varepsilon t)$ is the solution of system (2.12) averaged with respect to the phases α, γ; $\xi = (a, b, r, \psi, \theta)$, D = const. The proof can be carried out using Gronwall's lemma, on the basis of the standard change-of-variable procedure of the averging method [2], as well as the arithmetic lemma used to estimate the "small denominators" [15] that arise in setting up the substitution in question.

System (2.10) is a single-frequency system in the resonant case (2.11). Indeed, instead of α we introduce a new slow variable, namely a linear combination of the phases with integer coefficients:

$$\lambda=\alpha-i(i-j)^{-1}\gamma,\quad (i/j)\neq 1,\quad (i/j)\leq 2,\quad i,j>0 \tag{2.13}$$

System (2.10) assumes the form of a standard system with rotating phase:

$$x^{\cdot}=\varepsilon X(x,\ i(i-j)^{-1}\gamma+\lambda,\ \gamma)$$
$$\lambda^{\cdot}=\varepsilon Y(x,\ i(i-j)^{-1}\gamma+\lambda),\quad \gamma^{\cdot}=(C-A)A^{-1}r \qquad (2.14)$$

its right sides being $(2|i - j|\pi)$-periodic in γ. We set up the first-approximation system by averaging the right sides of system (2.14) with respect to the above period of variation of the argument γ. As a result we obtain the following system of equations for the slow variables:

$$a^{\cdot}=\varepsilon A^{-1}\mu_1^{*}-\varepsilon KC^{-1}r^{-1}b\cos\theta+\varepsilon KC^{-2}r^{-2}\sin\theta\mu_3^{**}$$
$$b^{\cdot}=\varepsilon A^{-1}\mu_2^{*}+\varepsilon KC^{-1}r^{-1}a\cos\theta-\varepsilon KC^{-2}r^{-2}\sin\theta\mu_3^{**}$$
$$r^{\cdot}=\varepsilon C^{-1}\mu_3^{*},\quad \psi^{\cdot}=\varepsilon KC^{-1}r^{-1},\quad \theta^{\cdot}=0,\quad \lambda^{\cdot}=-\varepsilon KC^{-1}r^{-1}\cos\theta$$
$$\mu_1^{*}(a,b,r,\psi,\theta,\lambda)=\frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|}(M_1^{\circ}\cos\gamma+M_2^{\circ}\sin\gamma)\,d\gamma$$
$$\mu_2^{*}(a,b,r,\psi,\theta,\lambda)=\frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|}(M_1^{\circ}\sin\gamma-M_2^{\circ}\cos\gamma)\,d\gamma,$$
$$\mu_3^{*}(a,b,r,\psi,\theta,\lambda)=\frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|}M_3^{\circ}\,d\gamma \qquad (2.15)$$
$$\mu_3^{**}(a,b,r,\psi,\theta,\lambda)=\frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|}M_3^{\circ}\sin\alpha\,d\gamma$$
$$\mu_3^{**}(a,b,r,\psi,\theta,\lambda)=\frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|}M_3^{\circ}\cos\alpha\,d\gamma$$

It is assumed that the variable α in the integrands is replaced by λ in accordance with (2.13). Note that the next-to-last equation in (2.15) has the solution $\theta = \theta_0$.

Solving averaged system (2.15) for perturbing moments of a particular form, we can determine the motion of the body in the resonant case with an error of order ε on a interval of time variation of order ε^{-1}. This can be substantiated in standard fashion [1, 2].

In what follows, we will employ the above technique to consider some specific examples of perturbed motion of a rigid body.

3. As an example of the technique, let us consider perturbed Lagrange motion with allowance for the moments acting on our rigid body from the environment. We will assume that the perturbing moments M_i $(i = 1, 2, 3)$ are linear-dissipative [16]:

$$M_1=-\varepsilon I_1 p,\quad M_2=-\varepsilon I_1 q,\quad M_3=-\varepsilon I_3 r,\quad I_1,\ I_3>0 \qquad (3.1)$$

Here I_1, I_3 are constant proportionality factors that depend on the properties of the medium and the shape of the body.

Let us write the perturbing moments with allowance for expressions (1.4) for p and q:

$$M_1=-\varepsilon^2 I_1 P,\quad M_2=-\varepsilon^2 I_1 Q,\quad M_3=-\varepsilon I_3 r \qquad (3.2)$$

In accordance with Sec. 2, for the nonresonant case we change over to new slow variables a, b, r, ψ, θ, and we obtain averaged system (2.12) of the form

$$a^{\cdot}=-\varepsilon I_1 A^{-1}a-\varepsilon KC^{-1}r^{-1}b\cos\theta,\quad b^{\cdot}=-\varepsilon I_1 A^{-1}b+\varepsilon KC^{-1}r^{-1}a\cos\theta$$
$$r^{\cdot}=-\varepsilon I_3 C^{-1}r,\quad \psi^{\cdot}=\varepsilon KC^{-1}r^{-1},\quad \theta^{\cdot}=0 \qquad (3.3)$$

Integrating the third equation in (3.3), we obtain (r_0 is the arbitrary initial value of the axial rotational velocity):

$$r = r_0 \exp(-\varepsilon I_3 C^{-1} t), \quad r_0 \neq 0 \tag{3.4}$$

Equation (3.3) for ψ' can be integrated with allowance for (3.4); it yields (ψ_0 is a constant equal to the initial value of the precession angle for $t = 0$):

$$\psi = \psi_0 + K I_3^{-1} r_0^{-1} [\exp(\varepsilon I_3 C^{-1} t) - 1] \tag{3.5}$$

In addition, as can be seen from (3.3), the angle of nutation maintains constant value $\theta = \theta_0$. Substituting (3.4) for r in the first two equations in (3.3), we obtain a system of the form

$$a' = -\varepsilon I_1 A^{-1} a - \varepsilon K C^{-1} r_0^{-1} \exp(\varepsilon I_3 C^{-1} t)\, b \cos\theta$$
$$b' = -\varepsilon I_1 A^{-1} b + \varepsilon K C^{-1} r_0^{-1} \exp(\varepsilon I_3 C^{-1} t)\, a \cos\theta$$

whose solution in accordance with [17] (p. 534) is described as follows:

$$\begin{aligned} a &= \exp(-\varepsilon I_1 A^{-1} t)(P_0 \cos\eta + Q_0 \sin\eta - K C^{-1} r_0^{-1} \sin\theta_0 \sin(\eta + \varphi_0)] \\ b &= \exp(-\varepsilon I_1 A^{-1} t)[P_0 \sin\eta - Q_0 \cos\eta + K C^{-1} r_0^{-1} \sin\theta_0 \cos(\eta + \varphi_0)] \\ \eta &= K I_3^{-1} r_0^{-1} \cos\theta_0 [\exp(\varepsilon I_3 C^{-1} t) - 1] \end{aligned} \tag{3.6}$$

As a result of substitution into expressions (2.5) and (1.4) for P, Q, p, q of the expressions for a and b from (3.6) and for r from (3.4), we obtain

$$\begin{aligned} p &= \exp(-\varepsilon I_1 A^{-1} t)[p_0 \cos(\gamma - \eta) - q_0 \sin(\gamma - \eta) + \\ &+ k C^{-1} r_0^{-1} \sin\theta_0 \sin(\gamma - \eta - \varphi_0)] + k C^{-1} r_0^{-1} \exp(\varepsilon I_3 C^{-1} t) \sin\theta_0 \sin\varphi \\ q &= \exp(-\varepsilon I_1 A^{-1} t)[p_0 \sin(\gamma - \eta) + q_0 \cos(\gamma - \eta) - \\ &- k C^{-1} r_0^{-1} \sin\theta_0 \cos(\gamma - \eta - \varphi_0)] + k C^{-1} r_0^{-1} \exp(\varepsilon I_3 C^{-1} t) \sin\theta_0 \cos\varphi \\ \gamma &= \frac{C}{I_3} \frac{C - A}{A} \frac{r_0}{\varepsilon} [1 - \exp(-\varepsilon I_3 C^{-1} t)], \quad p_0 = \varepsilon P_0, \quad q_0 = \varepsilon Q_0 \end{aligned} \tag{3.7}$$

Thus we have constructed the solution of the first-approximation system for the slow variables in the case of dissipative moment (3.1). Let us point out some qualitative features of motion in the case in question. The modulus of the axial rotational velocity r decreases monotonically in exponential fashion in accordance with (3.4). The increment of the precession angle $\psi - \psi_0$ increases slowly exponentially in accordance with (3.5). It follows from (3.6) that the slow variables a and b tend monotonically to zero exponentially.

In accordance with (3.7), the terms of the projections p and q that are due to the initial values p_0, q_0, attenuate exponentially. At the same time, projections p and q contain exponentially increasing terms that are proportional to the restoring moment k, with the result that the quantity $(p^2+q^2)^{1/2}$ grows exponentially.

If resonance relation (2.11) is satisfied, then averaging should be performed in accordance with scheme (2.15). In this case, all the integrals μ_1* from (2.15) coincide with the corresponding integrals μ_1 of (2.12). Therefore resonance in effect does not occur, and the resultant solution is suitable for describing motion for any ratio $C/A \neq 1$.

Note that we can similarly investigate a case that is more general than (3.1), namely that of a linear relationship between the dissipative moments and the angular rotational velocity: $\mathbf{M} = -\varepsilon I \cdot \boldsymbol{\omega}$. Here I is a tensor defined by the matrix

$$\begin{Vmatrix} I_1 & \varepsilon I_{12} & \varepsilon I_{13} \\ \varepsilon I_{21} & I_2 & \varepsilon I_{23} \\ \varepsilon I_{31} & \varepsilon I_{32} & I_3 \end{Vmatrix}$$

in which the cross terms are small as compared to the diagonal ones.

4. Let us consider motion of a rigid body in the Lagrange case under the action of a

small moment that is constant in the attached axes and is applied along the axis of symmetry. In this case the perturbing moments M_i ($i = 1, 2, 3$) have the form

$$M_1 = M_2 = 0, \quad M_3 = \varepsilon M_3^* = \text{const} \tag{4.1}$$

Changing over to new slow variables a, b, r, ψ, θ, we obtain an averaged system of type (2.12) in the nonresonant case:

$$\begin{gathered} a^{\cdot} = -\varepsilon K C^{-1} r^{-1} b \cos\theta, \quad b^{\cdot} = \varepsilon K C^{-1} r^{-1} a \cos\theta \\ r^{\cdot} = \varepsilon C^{-1} M_3^*, \quad \psi^{\cdot} = \varepsilon K C^{-1} r^{-1}, \quad \theta^{\cdot} = 0 \end{gathered} \tag{4.2}$$

Integrating the third equation in (4.2), we obtain

$$r = r_0 + \varepsilon C^{-1} M_3^* t \tag{4.3}$$

We substitute (4.3) into (4.2) and integrate the equation for ψ:

$$\psi = \psi_0 + K (M_3^*)^{-1} \ln |1 + \varepsilon C^{-1} M_3^* r_0^{-1} t| \tag{4.4}$$

Here ψ_0 and r_0 are arbitrary initial values of the precession angle and axial rotational velocity.

As follows from (4.2), the nutation angle θ does not change during the time of motion of the body $\theta = \theta_0$.

After replacing r by (4.3), the solution of the system consisting of the first two equations of (4.2) can be written as follows:

$$\begin{gathered} a = P_0 \cos\beta + Q_0 \sin\beta - K C^{-1} r_0^{-1} \sin\theta_0 \sin(\beta + \varphi_0) \\ b = P_0 \sin\beta - Q_0 \cos\beta + K C^{-1} r_0^{-1} \sin\theta_0 \cos(\beta + \varphi_0) \\ \beta = K (M_3^*)^{-1} \cos\theta_0 \ln |1 + \varepsilon C^{-1} r_0^{-1} M_3^* t| \end{gathered} \tag{4.5}$$

Substituting the resultant expressions for a and b from (4.5) and for r from (4.3) into (2.5) and (1.4), we obtain

$$\begin{gathered} p = p_0 \cos(\gamma - \beta) - q_0 \sin(\gamma - \beta) + k C^{-1} r_0^{-1} \sin\theta_0 \times \\ \times \sin(\gamma - \beta - \varphi_0) + k C^{-1} (r_0 + \varepsilon C^{-1} M_3^* t)^{-1} \sin\theta_0 \sin\varphi \\ q = p_0 \sin(\gamma - \beta) + q_0 \cos(\gamma - \beta) - k C^{-1} r_0^{-1} \sin\theta_0 \times \\ \times \cos(\gamma - \beta - \varphi_0) + k C^{-1} (r_0 + \varepsilon C^{-1} M_3^* t)^{-1} \sin\theta_0 \cos\varphi \\ \gamma = (C - A) A^{-1} ({}^1/_2 \varepsilon C^{-1} M_3^* t^2 + r_0 t), \quad p_0 = \varepsilon P_0, \quad q_0 = \varepsilon Q_0 \end{gathered} \tag{4.6}$$

In accordance with (4.3), the quantity $|r(\tau)|$, $\tau = \varepsilon t$ increases if the parameters r_0, M_3^* are of the same sign, and decreases if the signs are different. The precession angle ψ (4.4) contains a variable component whose modulus increases monotonically in both cases: in the first case it is bounded for finite $\tau \sim 1$, while in the second it tends to infinity as $\tau \to -(Cr_0/M_3^*)$; here $r \to 0$.

The variable β in (4.5), (4.6) varies analogously to ψ if $\theta_0 \neq \pm {}^1/_2\pi$, and it has the meaning of the oscillation phase. The oscillation frequency $(d\beta/dt) \sim r_0^{-1}$. The slow variables a, b are bounded 2π-periodic functions of β.

The components p and q of the angular velocity vector, in accordance with (4.6), contain bounded oscillating terms that are due to the nonzero initial data p_0, q_0, and also an analogous term that is due to the restoring moment (1.2). The oscillation frequency is determined by the derivative of the variable $(\gamma - \beta)$, which has the meaning of phase.

5. Let us briefly discuss the case of a heavy rigid body for which the ellipsoid of inertia relative to point O is close to being an ellipsoid of revolution, so that its principal moments of inertia have the form $A=A^0(1+\varepsilon\delta_1)$, $B=A^0(1+\varepsilon\delta_2)$, $C\neq A^0$. Here δ_1, δ_2 are dimensionless constants of order unity; A^0 is the characteristic value of the moments of inertia. In addition, the center of gravity of the body may be shifted relative to point O* lying on the principal axis of inertia, relative to which the moment is equal to C, by an amount of order ε. In this case the problem can be reduced to the one considered in Sec. 1 by introducing additional perturbing moments that satisfy condition (1.4). It turns out that $M_3 \sim \varepsilon^2$ in this case, so that $M_3{}^* = M_3{}^0 = 0$. Following (2.12) and (2.15), we obtain $\mu_3=\mu_3'=\mu_3''=0$, $\mu_3{}^*=\mu_3{}^{*'}=\mu_3{}^{*''}=0$.

Thus, the last three equations in (2.12) assume the form $r^{\cdot}=0$, $\psi^{\cdot}=\varepsilon KC^{-1}r^{-1}$, $\theta^{\cdot}=0$.

In the approximation under consideration, the kinematic Euler equations are not perturbed and the motion of the body comprises regular precession.

Note that, as follows from the first-approximation equations (2.12) and (2.15), when several perturbing moments of the form (1.4) are present their results are added up, and integrals (2.12) and (2.15) corresponding to these perturbations are represented as a sum of integrals for the individual perturbations.

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