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SOME PROBLEMS OF THE MOTION OF A SOLID WITH A MOVING MASS

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Some cases of motion of a free solid containing a moving internal mass are investigated. First an analysis is made of passive motion of a solid carrying a moving point mass connected to the solid by an elastic coupling, with square-law friction present. A similar problem for viscous friction was investigated in [1,2]. Then the problem of response-optimal stabilization of a free solid with a moving mass coupled viscoelastically to the solid is solved. It is assumed that rotation is braked by means of a control moment that is bounded in modulus; the magnitude of the constraint may vary in time. Similar problems involving controlled motion of a solid relative to the center of mass were examined, e.g., in [3-5], and elsewhere.

1. Consider free motion of a solid to which a point mass M is attached at some point O_1 that is stationary relative to the solid. It is assumed that, under relative motion, point m is acted upon by a restoring elastic force with rigidity coefficient c, and also by a drag proportional to the square of the velocity, namely square-law friction with coefficient μ. Then the vector equation of the relative motion of point m, in accordance with the procedure of [1], can be written as follows:

$$\lambda|\mathbf{r}'|\mathbf{r}'+\Omega^2\mathbf{r}=-\{\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{\rho})+\boldsymbol{\omega}'\times\boldsymbol{\rho}+ \\ +(1-m/M)[\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})+\boldsymbol{\omega}'\times\mathbf{r}+2\boldsymbol{\omega}\times\mathbf{r}'+\mathbf{r}'']\} \quad (1.1)$$

Here $\Omega^2=c/m$, $\lambda=\mu/m$, $\boldsymbol{\rho}$ is the radius vector of point O_1; r is the radius vector of point m relative to O_1; ω is the absolute angular velocity of the solid; M is the total mass of the solid and moving point; and the prime denotes the differentiation with respect to time t in the coordinate system associated with the solid. Equation (1.1) can be more conveniently considered in the system associated with the solid; then ρ is a constant vector and ω is some as yet unknown function of time.

Our problem is to investigate the motion of the system, i.e., to find vectors r and ω that describe it as functions of time for specified arbitrary initial conditions. It is not possible to find a solution of the problem in the general case. However, if we assume that the coupling coefficients λ and Ω are such that "free" motion of point m resulting from the initial deviations attenuates much more rapidly than the solid makes one revolution, then in this case the motion of the solid is similar to Euler-Poinsot motion, and the relative oscillations of the point caused by this motion will be small. If we take

$$\lambda=\Lambda\Omega^3,\quad \Omega\gg\omega\quad (\omega=|\boldsymbol{\omega}|),$$

then the "forced" motion of system (1.1) can be written approximately in the form of the expansion

$$\mathbf{r}=-\Omega^{-2}\mathbf{a}+\Lambda\Omega^{-3}|\mathbf{a}'|\mathbf{a}+O(\Omega^{-4}) \\ \mathbf{a}=\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{\rho})+\boldsymbol{\omega}'\times\boldsymbol{\rho} \quad (1.2)$$

As we mentioned, the prime denotes the rate of change in the coordinate system associated with the solid. Furthermore, we assume that the origin of this system is at point O, the center of inertia of the solid and mass m. Then the equation that determines the unknown vector $\boldsymbol{\omega}(t)$ can be found from the condition that the moment of momentum of the system be constant relative to O, and can be written as follows [1]:

$$I_O^* \cdot \omega' + (\omega \times I_O^* \cdot \omega) = -(\mathbf{k}' + \omega \times \mathbf{k}) \tag{1.3}$$

Here I_O^* is the tensor of inertia of the solid and mass m at point O_1 relative to the center of inertia O. We can provisionally call **k** the vector of the moment of momentum of the moving mass m; it vanishes if there is no internal motion, i.e., $\mathbf{r}'=0$, $\mathbf{r}=0$. In the general case, taking account of (1.2), vector **k** is given approximately by the following expression:

$$\mathbf{k} = m[\rho \times (\omega \times \mathbf{r} + \mathbf{r}') + \mathbf{r} \times (\omega \times \rho)] + O(\Omega^{-4}) \tag{1.4}$$

Here **r** is computed approximately in accordance with (1.2), while the derivative **r'**, which is expressed in terms of ω', can be found by using an expression that follows directly from (1.3):

$$\omega' = -(I_O^*)^{-1} \cdot \omega \times I_O^* \cdot \omega + O(\Omega^{-2}) \tag{1.5}$$

Thus, **r'** can be determined with the requisite degree of accuracy as a function of ω. The subsequent derivatives **r"**, ω" can be similarly determined. As a result, to determine the angular velocity vector ω from (1.3) on the basis of (1.4) and (1.5), we obtain the desired equation of the form [1]:

$$I_O^* \cdot \omega' + \omega \times (I_O^* \cdot \omega) = \Phi(\omega) + O(\Omega^{-4}) \tag{1.6}$$

Here $\Phi(\omega)$ is a polynomial that contains the fourth and eighth powers of vector ω; it consists of terms whose magnitudes are of order Ω^{-2} and Ω^{-3}. In the general case, the form of Φ is fairly cumbersome and we will not give it here. In the next section, we will set up a solution of the Cauchy problem for (1.6) in the particular case of axial symmetry.

2. Let us investigate the motion of a dynamically symmetrical solid that carries a moving point mass m that is connected to the solid at some point O_1 on the axis of symmetry. It is assumed that, under relative motion, point m is acted upon by an elastic restoring force and a drag that is proportional to the square of the velocity (see §1). We locate the origin of a Cartesian coordinate system associated with the body at the center of inertia O of the system consisting of solid and point mass at point O_1. We direct the unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of this system in such a way that $\mathbf{e}_3$ coincides with the axis of dynamic symmetry of the system. Then the radius vector ρ of O_1 is equal to $\rho = \rho \mathbf{e}_3$; to be specific, we assume that $\rho > 0$. In this coordinate system the inertia tensor I_O^* is diagonal [1]:

$$I_O^* = \mathrm{diag}\,(I, I, I_*) \tag{2.1}$$

The quantities I and I_* are called the equatorial and axial moments of inertia, respectively. Since $\omega_i = \omega \cdot \mathbf{e}_i$ $(i=1, 2, 3)$, the zero-approximation equation of type (1.5) for computing the derivatives with allowance for (2.1) can be written in scalar form:

$$\omega_1' = d\omega_2\omega_3, \quad \omega_2' = -d\omega_1\omega_3, \quad \omega_3' = 0 \quad (d = 1 - I_* I^{-1}) \tag{2.2}$$

To determine the right side of the equation of motion of type (1.6), we compute **a** and **a'** in expression (1.2) for **r**. Using (2.2), we find

$$\mathbf{a} = \rho\omega_3(\omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2) I_* I^{-1} - \rho\omega_\perp^2 \mathbf{e}_3, \quad \omega_\perp = (\omega_1^2 + \omega_2^2)^{1/2}$$
$$\mathbf{a}' = \rho\omega_3^2 I_* I^{-1} d(\omega_2\mathbf{e}_1 - \omega_1\mathbf{e}_2) \tag{2.3}$$

As a result, we obtain an explicit expression for **k** in terms of the variables $\omega_1, \omega_2, \omega_3$, with an error $O(\Omega^{-4})$:

$$\mathbf{k} = m\rho^2\Omega^{-2}[(2\omega_\perp^2 + I_*^2 I^{-2}\omega_3^2)(\omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2) + I_* I^{-1}\omega_\perp^2\omega_3\mathbf{e}_3] -$$
$$- m\rho^3\Lambda\Omega^{-3} I_*^3 I^{-3} d|d|\omega_\perp\omega_3^5(\omega_2\mathbf{e}_1 - \omega_1\mathbf{e}_2) \tag{2.4}$$

Furthermore, the desired expression for the derivative **k'** can be computed in a similar fashion using (2.2). The computational procedure is fairly simple, but it should be noted that, in view of (2.2), $\omega_\perp' = 0$.

Now if we project (1.6) onto the unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, we obtain the desired equations of motion:

$$\omega_1' - d\omega_2\omega_3 = A\omega_2\omega_3 + B\omega_1\omega_3^6, \quad \omega_1(0) = \omega_{10} \tag{2.5}$$

$$\omega_2'+d\omega_1\omega_3=-A\omega_1\omega_3+B\omega_2\omega_3^6, \quad \omega_2(0)=\omega_{20}$$
$$\omega_3'=-BI^2I_*^{-2}\omega_\perp^2\omega_3^5, \quad \omega_3(0)=\omega_{30} \tag{2.5}$$

Here we have introduced the following notation to simplify matters:

$$A=m\rho^2\Omega^{-2}I_*I^{-4}K^2, \quad K^2=I^2\omega_\perp^2+I_*^2\omega_3^2, \quad B=m\rho^3\Lambda\Omega^{-3}I_*^4I^{-5}d|d|\omega_\perp \tag{2.6}$$

Adding the equations in (2.5), multiplied by $I^2\omega_1$, $I^2\omega_2$, and $I_*^2\omega_3$, respectively, we can find the first integral of motion—the modulus of the kinetic moment $K=|\mathbf{K}|$:

$$K=K_0=\text{const}, \quad K_0^2=I^2\omega_{\perp 0}^2+I_*^2\omega_{30}^2 \tag{2.7}$$

System (2.5) can be further integrated. We employ the following procedure [1] to determine $\boldsymbol{\omega}$. We define the projections of vector K onto the principal central axes of inertia as follows:

$$I\omega_1=K\sin\theta\cos\varphi, \quad I\omega_2=K\sin\theta\sin\varphi, \quad I_*\omega_3=K\cos\theta \tag{2.8}$$

Here θ is the angle of nutation, while φ is the angle of pure rotation. Since, in accordance with (2.7), we have $K=\text{const}$, differentiating (2.8), we obtain, in view of (2.5) and taking account of expressions (2.6), the following differential equations for the spherical angles θ, φ:

$$\varphi'=\beta\cos\theta, \quad \theta'=\gamma\sin\theta|\sin\theta|\cos^5\theta \tag{2.9}$$

The coefficients β and γ in (2.9) are constant and are given by the following:

$$\beta=-(d+m\rho^2\Omega^{-2}I_*I^{-4}K^2)I_*^{-1}K, \quad \gamma=m\rho^3\Lambda\Omega^{-3}I_*^{-2}I^{-2}d|d|K^7 \tag{2.10}$$

In the particular cases of spherical symmetry ($d = 0$) or $\rho = 0$, it follows from (2.10) that the constant $\gamma = 0$, while the equations in (2.9) can be integrated in explicit form:

$$\theta=\theta_0, \quad \varphi=\beta t\cos\theta_0+\varphi_0, \quad \theta_0,\ \varphi_0=\text{const}$$

The components of the angular velocity $\omega_{1,2,3}$ can also be computed explicitly using (2.8):

$$\omega_1=\omega_\perp\cos\varphi, \quad \omega_2=\omega_\perp\sin\varphi, \quad \omega_3=\omega_{30} \quad (\omega_\perp=\omega_{\perp 0})$$

Now let us consider the general case $\gamma\neq 0$. Integration of the second equation in (2.9) leads to the relationship

$$2\sec^4\theta\,\mathrm{cosec}\,\theta+5[(\sec^2\theta-3)\,\mathrm{cosec}\,\theta+$$
$$+3\ln|\mathrm{tg}(\pi/4+\theta/2)|]=8\gamma t+\text{const} \tag{2.11}$$

To be specific, let us assume that θ_0 belongs to the interval $(0, \pi/2)$. On the basis of (2.10), the sign of γ is determined by the sign of the parameter d, i.e., the difference $I-I_*$. It follows from (2.9) and (2.11) that for $I>I_*$ (extended solid) the angle of nutation θ increases monotonically and tends to $\pi/2$ as $t\to\infty$, while $\omega_3\to 0$. For $I<I_*$ (oblate) body we find that θ decreases monotonically: $\theta\to 0$ as $t\to\infty$, while $\varphi'\to\beta=\text{const}$. Thus, the direction of the kinetic moment vector K in the coordinate system associated with the body tends to a stationary state, namely to the directions of the axes corresponding to the largest moments of inertia. If, in accordance with (2.11), $\theta(t)$ has been determined, then $\varphi(t)$ can be found by quadratures from the first equation in (2.9). By dividing the first equation in (2.9) by the second and then using quadratures, we obtain $\varphi(\theta)$, which, together with (2.11), yields an implicit solution of system (2.5) using (2.8). Note that the constant $\tau=|\gamma|^{-1}$ has the dimension of time and characterizes the rate at which the motion of the solid realigns itself, i.e., the rate of the angle of nutation θ: $\theta\to 0$ or $\theta\to\pi/2$. For the case of viscoelastic coupling between the solid and mass m [1,2], a similar time constant τ determines the time interval over which the angle of nutation decreases or increases by a factor of e in the linear approximation. In the problem under consideration with square-law friction for small θ, the angle of nutation tends to zero much more slowly, since $\theta'\sim-\tau^{-1}\theta^2$. In the principal (quadratic) term, the values of $\theta(t)$ are related by the expression $\theta(t-\tau)-\theta(t)=\pm\theta(t-\tau)\theta(t)$, while in the linear approximation $\theta=\text{const}$.

3. Consider controlled motion of a dynamically symmetrical solid and moving mass m coupled viscoelastically with coefficients δ and c of viscous friction and rigidity,

respectively. It is assumed that the point O_1 of attachment is on the axis of symmetry, while the values of the control-moment vector relative to the center of inertia O are bounded by a sphere and have the form

$$M_i = bu_i \quad (i=1,2,3), \qquad |\mathbf{u}| \leq 1 \quad (b_1 \geq b(t) \geq b_0 \geq 0) \tag{3.1}$$

It is assumed that $\Omega^2 \gg \nu\omega \gg \omega^2$, i.e., the natural oscillations of mass m attenuate much more rapidly than the solid makes one revolution [1]. Similarly to §1 and §2, the equations of controlled motion can be brought to the form

$$\begin{gathered} \omega_1' - d\omega_2\omega_3 = bI^{-1}u_1 + A\omega_2\omega_3 + N\omega_1\omega_3^4 \\ \omega_2' + d\omega_1\omega_3 = bI^{-1}u_2 - A\omega_1\omega_3 + N\omega_2\omega_3^4 \\ \omega_3' = bI_*^{-1}u_3 - NI^2I_*^{-2}\omega_\perp^2\omega_3^2, \quad \boldsymbol{\omega}(t_0) = \boldsymbol{\omega}_0 \end{gathered} \tag{3.2}$$

Here the coefficient A is determined in accordance with (2.6), while N is given by [1]

$$N = m\rho^2\nu\Omega^{-4}I_*^3I^{-4}d, \quad \nu = \delta/m, \quad \Omega^2 = c/m$$

Our problem is to find the response-optimal braking of the rotations of the system, i.e.,

$$\boldsymbol{\omega}(T) = 0, \quad T \to \min, \quad |\mathbf{u}| \leq 1 \tag{3.3}$$

We need to find an optimal control law, optimal phase trajectory, and minimum value of the functional. Note that similar problems of response-optimal stabilization were considered earlier for a solid without allowance for the possibility of motion of internal masses [3,4].

A solution of the braking problem can be set up on the basis of the sufficient optimality conditions of the dynamic programming method [6]. Using the functional Schwartz inequality [7] for $\mathbf{K}'$, we find that the synthesis of the optimal control has a fairly simple form: $\mathbf{u}^* = -\mathbf{K}K^{-1}$, while, in accordance with this law, the modulus of the kinetic moment K decreases to zero over a finite time T^*:

$$K(t, t_0, K_0) = K_0 - \int_{t_0}^{t} b(\tau)\,d\tau, \quad \int_{t_0}^{T} b(t)\,dt = K_0 \tag{3.4}$$

Since, in accordance with (3.1), we have $b \geq b_0 > 0$, the root of the second equation in (3.4), $T^* = T(K_0, t_0)$, exists and is unique, where $T^* \leq t_0 + K_0 b_0^{-1}$. We can establish directly by differentiation that $T(K, t)$ is the Bellman function of optimal control problem (3.1)-(3.3).

Substitution of the familiar expression for K into the third equation of (3.2) leads to a nonlinear equation in ω_3:

$$\omega_3' = -\omega_3(bK^{-1} + I_*^{-2}NK^2\omega_3^2 - N\omega_3^4) \tag{3.5}$$

By making the change of variable $\omega_3 = KR$, where R is an unknown function, we can bring (3.5) to a form with separable variables:

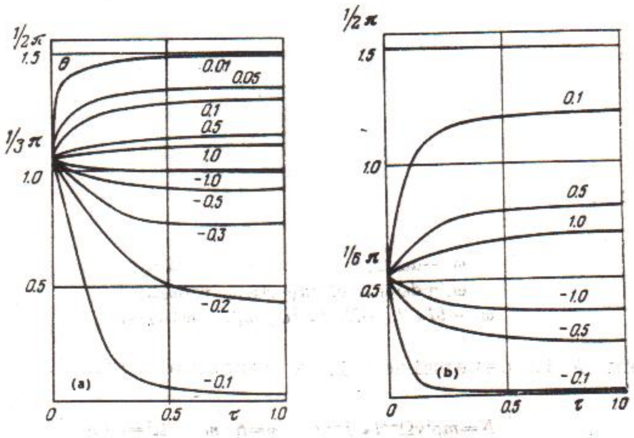
$$R^{2\prime} = -2NK^4R^4(I_*^{-2} - R^2) \tag{3.6}$$

Since, in accordance with (2.8), $I_*R = \cos\theta$, as a result of integration of (3.6) we find the relationship between the angle of nutation θ and the time t in implicit form:

$$\sec^2\theta - \sec^2\theta_0 + \ln \operatorname{tg}^2\theta \operatorname{tg}^{-2}\theta_0 = -\frac{2N}{I_*I^3}\int_{t_0}^{t} K^4(\tau)\,d\tau \tag{3.7}$$

Let us now consider for simplicity the case b = const; then we can find the relationship between t and θ in the following form $(t_0 = 0)$:

$$\begin{gathered} \tau = 1 - [1 + \sigma(\operatorname{tg}^2\theta_0 - \operatorname{tg}^2\theta + \ln \operatorname{tg}^2\theta_0 \operatorname{tg}^2\theta)]^{1/5}, \quad \tau = t/T^* \quad (0 \leq \tau \leq 1) \\ T^* = \frac{K_0}{b}, \quad \sigma = \frac{5}{2}\frac{bI_*I^4\Omega^4}{m\rho^2\nu dK_0^5} \quad (-\infty < \sigma < \infty) \end{gathered} \tag{3.8}$$



The accompanying figure (a and b) shows plots of the angle of nutation $\theta(\tau)$ for initial values $\theta_0=\pi/3,\ \pi/6$ and various values of the parameter σ, indicated next to the corresponding curves. In the approximation under consideration, we can draw the following qualitative conclusions: as $\sigma\to\pm 0$ the quantity $\theta(\tau)$ tends to a right angle and to zero, respectively; this is associated with the fact that the time t is unrestrictedly "compressed" (τ is slow or compressed time). As $\sigma\to\pm\infty$ we have $\theta(\tau)\to\theta_0$, since the angle of nutation θ does not vary markedly over the braking time T^* for the rotations of the solid. In the limit as $b_1\to 0$, formulas (3.7) and (3.8) coincide with those obtained in [1] for the passive-motion problem.

Now, on the basis of the known relationship between θ and t, we can readily obtain from (2.8), (3.4), (3.7), or (3.8) the time dependence of the axial angular velocity ω_3:

$$\omega_3(t)=K(t)\cos\theta(t) \qquad (0\leqslant t\leqslant T^*)$$

If this function is constructed, then for ω_1, ω_2 we obtain from (3.2) the following explicit expressions in the form of quadratures:

$$\omega_1=\omega_{10}KK_0^{-1}\exp\alpha(t)\cos\psi(t),\quad \omega_2=-\omega_{20}KK_0^{-1}\exp\alpha(t)\sin\psi(t)$$

$$\alpha(t)=N\int_{t_0}^{t}\omega_3^4(\tau)\,d\tau,\quad \psi(t)=\int_{t_0}^{t}[d+A(\tau)]\,\omega_3(\tau)\,d\tau \tag{3.9}$$

$$A(t)=m\rho^2\Omega^{-2}I_*I^{-4}K^2(t)$$

As follows from (3.9), the frequency $\psi'(T^*)=0$, while $\omega_\perp=\omega_{\perp 0}KK_0^{-1}\exp\alpha$, where $I^2\omega_\perp^2+I_*^2\omega_3^2=K^2$. Substitution of the resultant functions for the optimal phase trajectory into the expression for the synthesis of the control $\mathbf{u}^*$ yields the optimal programmed control. Thus, our problem of response-optimal stabilization of the system may be regarded as solved.

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REFERENCES

1. F. L. Chernous'ko, "Motion of a solid with moving internal masses," Izv. AN SSSR. MTT [Mechanics of Solids], no. 4, 1973.
2. D. D. Leshchenko, "Motion of a solid with a moving point mass," Izv. AN SSSR. MTT [Mechanics of Solids], no. 3, 1976.
3. L. D. Akulenko and Yu. R. Roshchin, "Response-optimal braking of the rotation of a solid by controls limited by an ellipsoid," Izv. AN SSSR. MTT [Mechanics of Solids], no. 1, 1977.
4. L. D. Alulenko and Yu. R. Roshchin, "Optimal control of the rotation of a solid by a low-thrust pivoted motor," Izv. AN SSSR. MTT [Mechanics of Solids], no. 5, 1977.
5. B. A. Smol'nikov, "Generalization of Euler case of motion of a solid," PMM, vol. 31, no. 4, 1967.
6. V. G. Boltyanskii, Mathematical Methods of Optimal Control [in Russian], Nauka Press, Moscow, 1969.
7. M. Athans and R. Falb, Optimal Control [Russian translation], Mashinostroenie, Moscow, 1968.

20 February 1978 Moscow, Odessa