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Dmytro Leshchenko
Odessa State Academy of Civil Engineering and Architecture

Mechanics of Solids
Vol. 32, No. 6, pp. 12–20, 1997

Izvestiya AN. MTT
No. 6, pp. 17–26, 1997

EVOLUTION OF ROTATION OF A TRIAXIAL BODY UNDER THE ACTION OF THE TORQUE DUE TO LIGHT PRESSURE

D. D. Leshchenko

(Received 19 March 1996)

A vast literature (e.g., see [1–7] and the bibliography therein) deals with the investigation of the motion of a satellite about the center of mass under the action of torques of various nature (gravitational, magnetic, light pressure, and the like). The evaluation [1] of the perturbing torques shows that the torque due to light pressure exerts a considerable effect on the space vehicles at altitudes exceeding 35000–40000 km over the Earth surface. We apply the averaging method to investigate the motion of a spacecraft with respect to the center of mass caused by light pressure. It is assumed that the spacecraft is nearly dynamically spherical and that its surface is a surface of revolution. The coefficient of the torque due to light pressure is approximated by a trigonometric polynomial.

1. BASIC ASSUMPTIONS AND STATEMENT OF THE PROBLEM

Consider the motion of a spacecraft (satellite) about its center of mass under the action of the torque due to light pressure. The center of mass of the spacecraft moves along an elliptic orbit around the Sun. Let us introduce three right-handed Cartesian reference frames centered at the center of mass of the satellite [1, 2]. The reference frame $OXYZ$ moves translationally so that the Y-axis is normal to the plane of the orbit, the Z-axis is codirected with the position vector of the perihelion of the orbit, and the X-axis is codirected with the velocity of the satellite center of mass at the perihelion. To define the direction of the angular momentum $\mathbf{L}$ of the satellite about the center of mass in the frame $OXYZ$, we introduce the angles ρ and σ, as is done in [1, 2, 4]. To construct the reference frame OL_1L_2L associated with the vector $\mathbf{L}$, we draw the axis L_1 in the plane OYL so that L_1 is perpendicular to $\mathbf{L}$ and forms an obtuse angle with the Y-axis. The axis L_2 complements the axes L_1 and L to a right-handed trihedral. The axes of the coordinate frame $Oxyz$ rigidly attached to the satellite coincide with the principal central axes of inertia of the satellite. The relative position of the principal central axes of inertia of the satellite and the axes L, L_1, and L_2 are defined by the Eulerian angles φ, ψ, and θ [1, 2, 4]. The direction cosines α_{ij} of the x-, y-, and z-axes in the OL_1L_2L reference frame are related to the Eulerian angles φ, ψ, and θ by well-known formulas [1].

We neglect the moments of all forces apart from those of the light pressure. We assume that the surface of the satellite is a surface of revolution with unit vector $\mathbf{k}$ of the symmetry axis pointing along the z-axis. In this case [1, 3, 5], the torque $\mathbf{M}$ due to the light pressure forces applied to the satellite is expressed by

$$\mathbf{M} = a_c(\varepsilon_s)\frac{R_0^2}{R^2}\mathbf{e}_r \times \mathbf{k}, \qquad a_c(\varepsilon_s)\frac{R_0^2}{R^2} = p_c S(\varepsilon_s) z_0'(\varepsilon_s), \quad p_c = \frac{E_0}{c}\frac{R_0^2}{R^2}, \tag{1.1}$$

where $\mathbf{e}_r$ is the unit vector of the position of the satellite center of mass; ε_s is the angle between the vectors $\mathbf{e}_r$ and $\mathbf{k}$ defined so that $|\mathbf{e}_r \times \mathbf{k}| = \sin\varepsilon_s$; R is the current distance between the center of the Sun and the center of mass of the satellite; R_0 is a fixed value of the variable R, for instance, at the initial time instant; $a_c(\varepsilon_s)$ is the coefficient of the torque due to the light pressure; S is the area of the shadow on the plane normal to the light flux; z_0' is the distance between the center of mass of the satellite and the center of pressure; p_c is the light pressure at the distance R from the center of the Sun; c is the velocity of light; E_0 is the magnitude of the radiant energy flux at the distance R_0 from the center of the Sun. If R_0 is the radius of the Earth orbit, then $p_{c0} = 4.64 \cdot 10^{-6}\,\mathrm{N/m^2}$.

In what follows, we assume that $a_c = a_c(\cos\varepsilon_s)$ [1] and approximate the function a_c by polynomials in $\cos\varepsilon_s$. The light pressure torque has a force function depending only on the orientation of the symmetry axis of the body [1]. We represent the function $a_c(\cos\varepsilon_s)$ in the form

$$a_c = a_0 + a_1\cos\varepsilon_s + \cdots + a_N\cos^N\varepsilon_s. \tag{1.2}$$

Since the light pressure torque has a force function, the equations of the perturbed motion of the satellite can be represented in the variables L, ρ, σ, φ, ψ, and θ in the form [2, 4]

$$\begin{aligned}
&\dot\sigma = \frac{1}{L\sin\rho}\frac{\partial U}{\partial\rho}, \quad \dot\rho = -\frac{1}{L\sin\rho}\frac{\partial U}{\partial\sigma} + \frac{\cot\rho}{L}\frac{\partial U}{\partial\psi}, \quad \dot L = \frac{\partial U}{\partial\psi},\\
&\dot\theta = L\sin\theta\sin\varphi\cos\varphi\left(\frac{1}{A}-\frac{1}{B}\right) - \frac{1}{L\sin\theta}\frac{\partial U}{\partial\varphi} + \frac{\cot\theta}{L}\frac{\partial U}{\partial\psi},\\
&\dot\varphi = L\cos\theta\left(\frac{1}{C}-\frac{\sin^2\varphi}{A}-\frac{\cos^2\varphi}{B}\right) + \frac{1}{L\sin\theta}\frac{\partial U}{\partial\theta},\\
&\dot\psi = L\left(\frac{\sin^2\varphi}{A}+\frac{\cos^2\varphi}{B}\right) - \frac{1}{L}\left(\cot\rho\frac{\partial U}{\partial\rho} + \cot\theta\frac{\partial U}{\partial\theta}\right).
\end{aligned} \tag{1.3}$$

The force function U depends on time (via the true anomaly $\nu(t)$) and the direction cosines α_3, β_3, and γ_3 of the z-axis in the reference frame $OXYZ$; it has the form $U = U(\nu(t), \alpha_3, \beta_3, \gamma_3)$.

System (1.3) must be completed by the equation

$$\frac{d\nu}{dt} = \omega_0\frac{(1+e\cos\nu)^2}{\sqrt{(1-e^2)^3}}, \qquad \omega_0 = \frac{2\pi}{T_0} = \sqrt{\frac{\kappa(1-e^2)^3}{P^3}}, \tag{1.4}$$

describing the variation of the true anomaly in time. Here ω_0 is the average angular velocity of motion of the center of mass along the elliptic orbit; T_0 is the period of revolution of the satellite; e and P are the eccentricity and the focal parameter of the orbit, respectively; κ is the product of the gravitational constant by the mass of the Sun.

The torque (1.1) corresponds to the force function

$$U(\cos\varepsilon_s) = -\frac{R_0^2}{R^2}\int a_c(\cos\varepsilon_s)\,d(\cos\varepsilon_s).$$

First, let us consider the case

$$a_c(\cos\varepsilon_s) = a_n\cos^n\varepsilon_s. \tag{1.5}$$

Then the force function has the form

$$U_n(\cos\varepsilon_s) = -\frac{a_nR_0^2}{(n+1)R^2}\cos^{n+1}\varepsilon_s, \qquad \cos\varepsilon_s = \gamma_3\cos\nu + \alpha_3\sin\nu. \tag{1.6}$$

The direction cosines α_3 and γ_3 can be expressed via ρ, σ, θ, and ψ according to well-known formulas [1].

Assume that the principal central moments of inertia of the satellite are close to each other and can be represented in the form

$$A = J_0 + \varepsilon A', \quad B = J_0 + \varepsilon B', \quad C = J_0 + \varepsilon C', \tag{1.7}$$

where ε is a small parameter, $0 < \varepsilon \ll 1$. Assume also that $a_0 \sim \varepsilon, a_1 \sim \varepsilon, \ldots, a_N \sim \varepsilon$, that is, both the light pressure torques and the gyroscopic torques are of the order of ε. It follows from (1.6) that $U \sim \varepsilon$. We investigate the solution of system (1.3), (1.4) for small ε on a large time interval $t \sim \varepsilon^{-1}$. The error of the averaged solution for the slow variables is $O(\varepsilon)$ on the time interval in which the body performs about ε^{-1} revolutions. We perform the averaging with respect to ψ and ν independently, as for nonresonance cases [2].

2. TRANSFORMATION OF THE EXPRESSION FOR THE FORCE FUNCTION. AVERAGING AND CONSTRUCTION OF THE SYSTEM OF THE FIRST APPROXIMATION

Consider the nonperturbed motion corresponding to $\varepsilon = 0$. In this case, Eqs. (1.3) and (1.4) describe the motion of a spherically symmetric body, the light pressure torque (1.1) being equal to zero. For this case, it follows from system (1.3) that the variables σ, ρ, L, θ, and φ are constant, whereas the variable ψ is a linear function of time

$$\psi = \frac{L}{J_0}t + \psi_0, \qquad \psi_0 = \text{const}, \tag{2.1}$$

corresponding to the uniform rotation of the satellite about the vector $\mathbf{L}$ of angular momentum, which moves translationally. For small $\varepsilon \neq 0$, in system (1.3), (1.4), where A, B, and C are related by (1.7), the variables σ, ρ, L, θ, and φ are slow variables and ψ and ν are fast variables. To obtain the solution in the first approximation, it suffices to substitute $\psi(t)$ given by (2.1) and $\nu(t)$ from the solution of Eq. (1.4) into the right-hand sides of (1.3) and then average the right-hand sides with respect to time. Suppose that $m_1\omega_0 + n_1 L/J_0 \neq 0$ for any integer m_1 and n_1, that is, the frequencies ω_0 and L/J_0 are not in resonance. In this case, averaging over time can be replaced by independent averaging over the variables ψ and ν according to a special scheme [2]. By virtue of (1.4), averaging over time for functions depending on ν is reduced to averaging over ν as follows:

$$M_t\{f(\nu)\} = \frac{1}{T}\int_0^T f(\nu)\,dt = \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-e^2)^{3/2} f(\nu)\,d\nu}{(1+e\cos\nu)^2} = (1-e^2)^{3/2} M_\nu\left\{\frac{f(\nu)}{(1+e\cos\nu)^2}\right\}. \tag{2.2}$$

The factor $\cos^{n+1}\varepsilon_s$ in the expression (1.6) for the force function can be represented in the form

$$\cos^{n+1}\varepsilon_s = (d + g\cos v)^{n+1}, \tag{2.3}$$

where

$$d = \cos\theta\sin\rho\cos(\sigma-\nu), \qquad v = \psi - \chi, \qquad g = \sqrt{\sin^2\theta\,[\sin^2(\sigma-\nu)\sin^2\rho + \cos^2\rho]},$$
$$\cos\chi = \frac{\sin\theta\sin(\sigma-\nu)}{\sqrt{\sin^2\theta\,[\sin^2(\sigma-\nu)\sin^2\rho+\cos^2\rho]}}, \qquad \sin\chi = \frac{\sin\theta\cos\rho\cos(\sigma-\nu)}{\sqrt{\sin^2\theta\,[\sin^2(\sigma-\nu)\sin^2\rho+\cos^2\rho]}}.$$

Using the binomial theorem, we can express the right-hand side of (2.3) as

$$(d + g\cos v)^{n+1} = \sum_{k=0}^{n+1} C_{n+1}^k \cos^k v\,(g^k d^{n+1-k}). \tag{2.4}$$

Using the well-known expressions for the direction cosines α_3, β_3, and γ_3 of the z-axis in the reference frame $OXYZ$ [1], we can obtain the average of the force function with respect to ψ. To this end, consider the integral

$$\frac{1}{2\pi}\int_0^{2\pi}(d+g\cos v)^{n+1}\,dv = \sum_{k=0}^{n+1} C_{n+1}^k g^k d^{n+1-k} I_k, \tag{2.5}$$

where

$$I_k = \frac{1}{2\pi}\int_0^{2\pi}\cos^k v\,dv, \quad I_{2m+1} = 0, \quad I_{2m} = \frac{(2m-1)!!}{(2m)!!}.$$

Thus, relation (2.5) can be represented as

$$\frac{1}{2\pi}\int_0^{2\pi}(d+g\cos v)^{n+1}\,dv = \sum_{m=0}^{E[(n+1)/2]} C_{n+1}^{2m}(g^{2m}d^{n+1-2m})\frac{(2m-1)!!}{(2m)!!}, \tag{2.6}$$

where $E[z]$ is the integer part of z. Averaging of the force function over ψ with allowance for (2.6) yields

$$U_n = -\frac{a_n R_0^2}{(n+1)R^2}\sum_{m=0}^{E[(n+1)/2]} C_{n+1}^{2m}(g^{2m}d^{n+1-2m})\frac{(2m-1)!!}{(2m)!!}. \tag{2.7}$$

To simplify the notation, we use the same letters for the original and averaged variables.

Let us now carry out averaging over ν according to (2.2). Note that since the center of mass of the satellite moves along an elliptic orbit, we have $R = P/(1 + e\cos\nu)$, and hence, according to (2.2) and (2.7), the expressions $(1 + e\cos\nu)^2$ in the formula for U_n are canceled.

Set $u = \sigma - \nu$. Then the variable d in Eq. (2.3) can be expressed as $d = h\cos u$, where $h = \cos\theta\sin\rho$. Let us represent the expression for g^{2m} in (2.7) in the form

$$g^{2m} = \{\sin^2\theta\,[\sin^2(\sigma-\nu)\sin^2\rho + \cos^2\rho]\}^m = (b + q\sin^2 u)^m, \qquad b = \sin^2\theta\cos^2\rho, \quad q = \sin^2\theta\sin^2\rho.$$

By once more applying the binomial theorem, we obtain

$$(b + q\sin^2 u)^m = \sum_{k=0}^{m} C_m^k (\sin u)^{2k}(q^k b^{m-k}).$$

For the second averaging with respect to the variable $n = \sigma - \nu$, consider the integral

$$\frac{1}{2\pi}\int_0^{2\pi}(b + q\sin^2 u)^m (h\cos u)^{n-2m+1}\,du = \sum_{k=0}^{m} C_m^k (q^k b^{m-k})\frac{1}{2\pi}\int_0^{2\pi}(\sin u)^{2k}(h\cos u)^{n+1-2m}\,du$$
$$= \sum_{k=0}^{m} h^{n+1-2m} C_m^k (q^k b^{m-k})\frac{1}{2\pi}\int_0^{2\pi}(\sin u)^{2k}(\cos u)^{n+1-2m}\,du.$$

This integral can be expressed in an explicit form [8]; for $n = 2l$, it is zero.

Let n be odd, i.e., $n = 2l + 1$. Then we have

$$k = 0, 1, \ldots, m; \quad m = 0, 1, \ldots, E[(n+1)/2] = l + 1.$$

Averaging the expression (2.7) with respect to $u = \sigma - \nu$ yields

$$U_{2l+1} = -\delta_l \sum_{m=0}^{l+1}\sum_{k=0}^{m} A_{lmk}(\cos\theta)^{2(l+1-m)}(\sin\theta)^{2m}(\sin\rho)^{2(l+1-m+k)}(\cos\rho)^{2(m-k)}, \tag{2.8}$$

where

$$\delta_l = \frac{a_{2l+1}R_0^2(1-e^2)^{3/2}}{2(l+1)P^2}, \qquad A_{lmk} = C_{2(l+1)}^{2m} C_m^k \frac{(2m-1)!!\,(2k-1)!!\,[2(l+1-m)-1]!!}{(2m)!!\,[2(k+l+1-m)]!!}.$$

The force function for the coefficient of the light pressure torque of the form (1.2) is represented as

$$U(\theta,\rho) = \sum_{l=0}^{Q} U_{2l+1}(\theta,\rho), \qquad Q = E[(N-1)/2]. \tag{2.9}$$

Thus, in the first approximation, the coefficient (1.2) of the light pressure torque has the form

$$a_c \sim \tilde{a}_c = \sum_{l=0}^{Q} a_{2l+1}(\cos\varepsilon_s)^{2l+1}, \tag{2.10}$$

since the even harmonics vanish on averaging.

By calculating the partial derivatives $\partial U/\partial\rho$ and $\partial U/\partial\theta$ of the function (2.8) and taking into account the fact that $\partial U/\partial\sigma = \partial U/\partial\psi = \partial U/\partial\varphi \equiv 0$, we arrive at the averaged system of the first approximation in the form

$$\begin{aligned}
\dot\sigma &= -\frac{2\delta_l}{L\sin\rho}\frac{\partial U}{\partial\rho}, \quad \dot\rho = 0, \quad \dot L = 0,\\
\dot\theta &= L\sin\theta\sin\varphi\cos\varphi\left(\frac{1}{A} - \frac{1}{B}\right),\\
\dot\varphi &= L\cos\theta\left(\frac{1}{C} - \frac{\sin^2\varphi}{A} - \frac{\cos^2\varphi}{B}\right) - \frac{2\delta_l}{L\sin\theta}\frac{\partial U}{\partial\theta},
\end{aligned} \tag{2.11}$$

where

$$\frac{\partial U}{\partial \rho} = \sum_{l=0}^{Q}\sum_{m=0}^{l+1}\sum_{k=0}^{m} A_{lmk}(\cos\theta)^{2(l+1-m)}(\sin\theta)^{2m}(\sin\rho)^{2(l-m+k)+1}(\cos\rho)^{2(m-k)-1}[(l+1)\cos^2\rho + k - m],$$

$$\frac{\partial U}{\partial \theta} = \sum_{l=0}^{Q}\sum_{m=0}^{l+1}\sum_{k=0}^{m} A_{lmk}(\sin\rho)^{2(l+1-m+k)}(\cos\rho)^{2(m-k)}(\cos\theta)^{2(l-m)+1}(\sin\theta)^{2m-1}[m - (l+1)\sin^2\theta].$$

Here the coefficients δ_l and A_{lmk} are defined in (2.8). Note that the coefficients a_{2l} of even powers of $\cos\varepsilon_s$ in Eq. (1.2) vanish on averaging. In what follows, we investigate system (2.11). The angular momentum is constant in magnitude and inclined at a constant angle to the normal to the plane of the orbit. Consider Eqs. (2.11) for the nutation angle θ and the angle of proper rotation φ. These equations describe the motion of the vector **L** of the angular momentum relative to the body.

3. INVESTIGATION OF THE EQUATIONS FOR θ AND φ

The equations for θ and φ in system (2.11) can be reduced to the form (in the slow time ξ)

$$\theta' = \sin\theta\sin\varphi\cos\varphi,$$
$$\varphi' = \cos\theta(\mu - \sin^2\varphi) - \frac{2\delta_l}{BL_0^2\sin\theta}\frac{\partial U(\theta,\rho_0)}{\partial\theta}, \tag{3.1}$$

where

$$\xi = L_0\beta t, \quad \mu = -\frac{\gamma}{\beta}, \quad \beta = \frac{1}{A} - \frac{1}{B}, \quad \gamma = \frac{1}{B} - \frac{1}{C}.$$

Here the coefficient δ_l is defined in (2.8) and the derivative $\partial U(\theta,\rho_0)/\partial\theta$ is given in (2.11); L_0 and ρ_0 are the initial values of the variables L and ρ, respectively. With allowance for relations (1.7) and the assumption that $a_{2l+1} \sim \varepsilon$ $(l = 0, \ldots, Q)$, we have $\beta, \gamma, \delta_l \sim \varepsilon$. One can verify by substitution that system (3.1) has the first integral

$$c = \sin^2\theta(\mu - \sin^2\varphi) - \frac{4\delta_l}{\beta L_0^2} f(\theta,\rho_0), \tag{3.2}$$

where

$$f(\theta,\rho_0) = \sum_{l=0}^{Q}\sum_{m=0}^{l+1}\sum_{k=0}^{m} A_{lmk}\left[\frac{m}{2}\sum_{i=0}^{l-m} C_{l-m}^{i}\frac{(-1)^i(\sin\theta)^{2(m+i)}}{m+i} - \frac{l+1}{2}\sum_{i=0}^{l-m}\frac{(-1)^i(\sin\theta)^{2(m+i+1)}}{m+i+1}\right](\sin\rho_0)^{2(l+1-m+k)+1}(\cos\rho_0)^{2(m-k)}.$$

It is convenient to represent the first integral (3.2) in the form

$$c = \sin^2\theta(\mu - \sin^2\varphi) - \frac{4\delta_l}{\beta L_0^2} F(\theta,\rho_0), \tag{3.3}$$

where

$$F(\theta,\rho_0) = \sum_{l=0}^{Q}\Bigg\{\left[-C_{2(l+1)}^{0}\frac{l+1}{2}\sum_{i=0}^{l} C_l^i\frac{(-1)^i(\sin\theta)^{2i+1}}{i+1}\right](\sin\rho_0)^{2(l+1)}\frac{[2(l+1)-1]!!}{[2(l+1)]!!}$$
$$+\sum_{m=1}^{l}\sum_{k=0}^{m} A_{lmk}\left[\frac{m}{2}\sum_{i=0}^{l-m}\frac{(-1)^i(\sin\theta)^{2(m+i)}}{m+i} - \frac{l+1}{2}\sum_{i=0}^{l-m} C_{l-m}^{i}\frac{(-1)^i(\sin\theta)^{2(m+i+1)}}{m+i+1}\right](\sin\rho_0)^{2(l+1-m+k)}(\cos\rho_0)^{2(m-k)}$$
$$+\frac{[2(l+1)-1]!!}{[2(l+1)]!!}\frac{(\sin\theta)^{2(l+1)}}{2}\sum_{k=0}^{l+1} C_{l+1}^{k}(\sin\rho_0)^{2k}(\cos\rho_0)^{2(l+1-k)}\frac{(2k-1)!!}{(2k)!!}\Bigg\}.$$

For the case where the light pressure torque does not act on the system, i.e., for $a_{2l+1} = 0$ ($l = 0, \ldots, Q$), system (3.1) can be reduced to the form

$$\theta' = \sin\theta \sin\varphi \cos\varphi, \quad \varphi' = \cos\theta\,(\mu - \sin^2\varphi). \tag{3.4}$$

For the case $n = 1$ ($l = 0$) [7], the first integral (3.3) is expressed as

$$c = \sin^2\theta\,(\mu^* - \sin^2\varphi) = \sin^2\theta_0\,(\mu^* - \sin^2\varphi_0) = \text{const}, \tag{3.5}$$

where

$$\mu^* = \frac{\alpha - \gamma}{\beta}, \quad \alpha = -\frac{1}{2}\frac{a_1 R_0^2}{P^2 L_0^2}(1 - e^2)^{3/2}(1 - \tfrac{3}{2}\sin^2\rho_0).$$

The expression (3.5) for μ^* coincides with the corresponding expression in [7] up to a factor of $\frac{1}{2}$. This is accounted for by the fact that the function $a_c(\cos\varepsilon_s)$ is represented in [7] in the form $a_c = a_0 + 2a_1\cos\varepsilon_s$. As was already mentioned, the term a_0 vanishes on averaging.

4. A SPECIAL CASE

Consider the function $a_c(\cos\varepsilon_s)$ of the form

$$a_c = \sum_{k=0}^{Q} a_{2k}\cos^{2k}\varepsilon_s + a_3\cos^3\varepsilon_s. \tag{4.1}$$

In this case, the equations for θ and φ in system (3.1) become (only the terms generated by the coefficient a_3 are retained)

$$\theta' = \sin\theta\sin\varphi\cos\varphi, \quad \varphi' = \cos\theta\,(\mu - \sin^2\varphi - \alpha\sin^2\theta), \qquad (\cdots)' = \frac{d}{d\xi}(\cdots), \tag{4.2}$$

where

$$\mu = -\frac{\gamma}{\beta} - \alpha\beta s, \quad \alpha = \frac{3}{64}\frac{a_3 R_0^2}{P^2 L_0^2}(1 - e^2)^{3/2}(8 - 40\sin^2\rho_0 + 35\sin^4\rho_0), \quad s = \frac{4\sin^2\rho_0\,(4 - 5\sin^2\rho_0)}{8 - 40\sin^2\rho_0 + 35\sin^4\rho_0}.$$

The variables ξ, β, and γ are defined in (3.1). With allowance for relation (1.7) and the assumption that $a_3 \sim \varepsilon$, we have $\beta, \alpha, \gamma \sim \varepsilon$.

System (4.2) has the first integral

$$c = \sin^2\theta\left(\mu - \sin^2\varphi - \tfrac{1}{2}\alpha\sin^2\theta\right) = \sin^2\theta_0\left(\mu - \sin^2\varphi_0 - \tfrac{1}{2}\alpha\sin^2\theta_0\right) = \text{const}. \tag{4.3}$$

This integral can be found either directly or from the expression (3.3) for the first integral for the case in which the function a_c is approximated by an arbitrary trigonometric polynomial. In the latter case, we must set $n = 3$ ($l = 1$) in Eq. (3.3).

Let us qualitatively investigate the phase plane $\theta\varphi$ of system (4.2) with the first integral (4.3). In this system, the variables θ and φ range in the intervals $0 \le \theta \le \pi$ and $0 \le \varphi \le 2\pi$, respectively, and the parameter μ can assume any value ($-\infty < \mu < +\infty$) depending on the relationship between the moments of inertia. The domain D of admissible values of the parameters (α, μ) is shown in Fig. 1.

We equate the right-hand sides of system (4.2) with zero to determine the critical points of this system. In doing so, we can separate four cases.

Case 1. $\cos\theta = 0$, $\theta = \pm\frac{1}{2}\pi$; $\varphi = 0$, $\pi \pm \frac{1}{2}\pi$. These critical points exist for any (μ, α).

Case 2. $\sin\theta = 0$, $\theta = 0, \pi$; $\mu - \sin^2\varphi = 0$, $0 \le \mu \le 1$, $\varphi = \pm\arcsin\sqrt{\mu}$, $\varphi = \pm\arcsin\sqrt{\mu} + \pi$. These critical points exist in the strip $0 \le \mu \le 1$.

Case 3. $\varphi = 0, \pi$; $\mu - \alpha\sin^2\theta = 0$, $0 < \mu/\alpha \le 1$, $\sin\theta = \pm\sqrt{\mu/\alpha}$, $\theta = \pm\arcsin\sqrt{\mu/\alpha}$, $\theta = \pm\arcsin\sqrt{\mu/\alpha} + \pi$. These critical points exist in the interior of the angles hatched with horizontal lines in Fig. 1, i.e., for $\mu \le \alpha$ if $\mu > 0$ and $\alpha > 0$ or $\mu \ge \alpha$ if $\mu < 0$ and $\alpha < 0$.

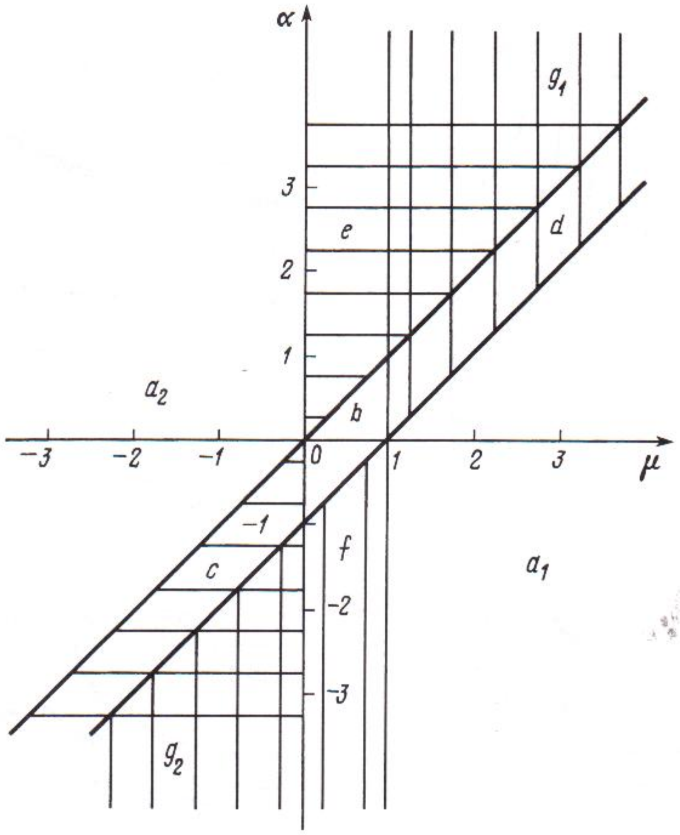

Fig. 1

Case 4. $\varphi = \pm\frac{1}{2}\pi$, $\mu - 1 - \alpha\sin^2\theta = 0$, $0 \le (\mu-1)/\alpha \le 1$, $\theta = \pm\arcsin\sqrt{(\mu-1)/\alpha}$, $\theta = \pm\arcsin\sqrt{(\mu-1)/\alpha} + \pi$. These critical points exist in the interior of the angles hatched with vertical lines, i.e., for $\mu - 1 \le \alpha$ if $\mu - 1 > 0$ and $\alpha > 0$ or $\mu - 1 \ge \alpha$ if $\mu - 1 < 0$ and $\alpha < 0$.

We can consider various characteristic cases with regard to the choice of the parameters μ and α. These cases, identified by letters a through g, correspond to geometric images in Fig. 1 as follows:

a_1 and a_2 correspond to the unhatched parts of the $\mu\alpha$-plane outside the angles;
b corresponds to the unhatched parallelogram in the central part;
c corresponds to the half-strip between the inclined lines hatched with horizontal lines;
d corresponds to the half-strip between the inclined lines hatched with vertical lines;
e corresponds to the "vertical" half-strip on the first quadrant of the $\mu\alpha$-plane hatched with horizontal lines;
f corresponds to the "vertical" half-strip on the fourth quadrant of the $\mu\alpha$-plane hatched with vertical lines;
g_1 and g_2 correspond to the double-hatched angles in the first and the third quadrants of the $\alpha\mu$-plane, respectively.

Figures 2–5 present the phase portraits of the averaged system constructed numerically for the above cases. It follows from Eq. (4.3) that all phase trajectories are symmetric with respect to the lines $\theta = \frac{1}{2}\pi$ and $\varphi = \frac{1}{2}\pi$. Therefore, it suffices to depict a quadrant of the phase portrait for $0 \le \theta \le \frac{1}{2}\pi$, $0 \le \varphi \le \frac{1}{2}\pi$.

In Fig. 2, the family of phase trajectories of the averaged system in the $\theta\varphi$-plane is presented for $\mu = -5$ and $\alpha = -2$ (case a_2). These curves correspond to the motion in which the system oscillates in the angle θ and oscillates (inside the separatrix) or rotates (outside the separatrix) in the angle φ. For case a_1 ($\mu = 5$, $\alpha = 2$), the behavior of the phase curves is similar. For case a_2, the system has a center-type critical point $\left(\frac{1}{2}\pi,\ \frac{1}{2}\pi\right)$ and a saddle-type point $\left(\frac{1}{2}\pi,\ 0\right)$; for case a_2, the point $\left(\frac{1}{2}\pi,\ 0\right)$ is a center and $\left(\frac{1}{2}\pi,\ \frac{1}{2}\pi\right)$ is a saddle.

The phase trajectories for $\mu = 0.8$ and $\alpha = 0.5$ (case b) are shown in Fig. 3. The critical points are $\left(\frac{1}{2}\pi,\ 0\right)$, $\left(\frac{1}{2}\pi,\ \frac{1}{2}\pi\right)$, and (0, 0.1071). For $\mu = 0.8$ and $\alpha = 3$ (case e), the phase curves describe oscillations in the variable θ and oscillations or rotations divided by a separatrix in the variable φ (see Fig. 4). The critical points for this case are $\left(\frac{1}{2}\pi,\ \frac{1}{2}\pi\right)$, $\left(\frac{1}{2}\pi,\ 0\right)$, (0, 1.1071), and (0.5426, 0).

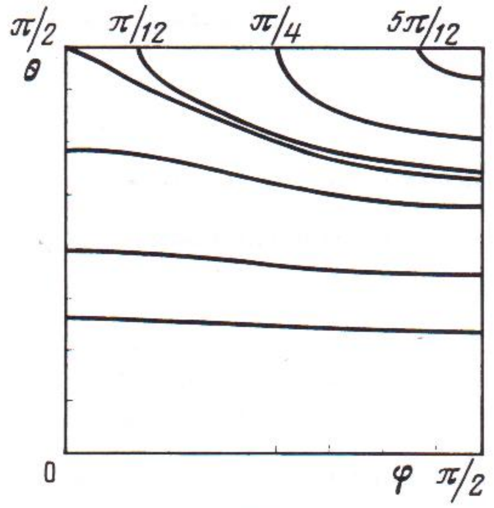

Fig. 2

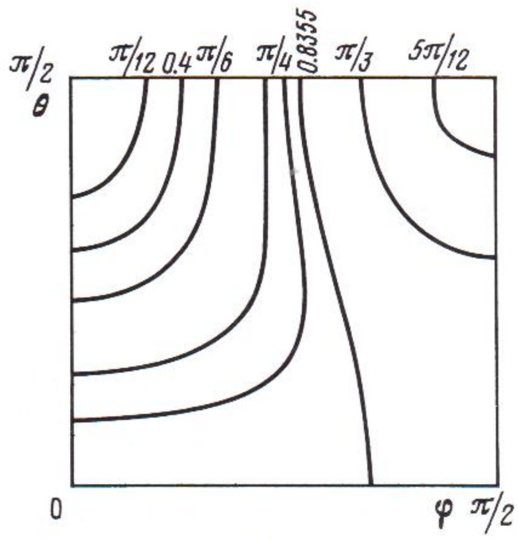

Fig. 3

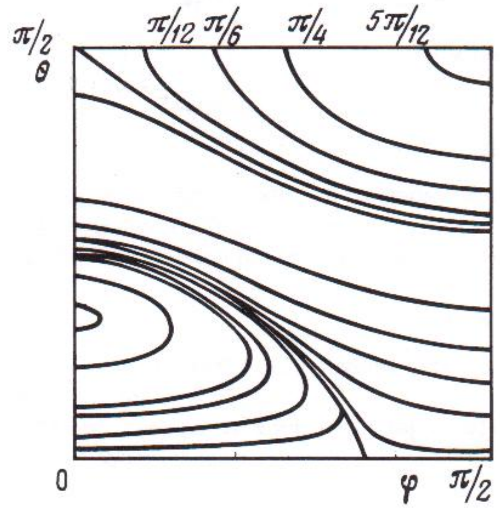

Fig. 4

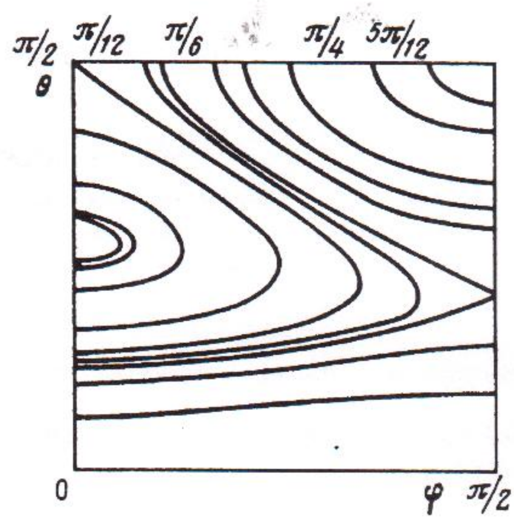

Fig. 5

The phase trajectories demonstrate a similar behavior for case f ($\mu = 0.8$, $\alpha = -1.5$). The critical points in this case are $\left(\frac{1}{2}\pi, 0\right)$, $\left(\frac{1}{2}\pi, \frac{1}{2}\pi\right)$, $(0, 1.1071)$, and $\left(0.3738, \frac{1}{2}\pi\right)$.

Figure 5 presents the phase trajectories for $\mu = 3$ and $\alpha = 5$ (case g_1). These trajectories describe oscillations in the variable θ. In the variable φ, oscillations occur inside and above the separatrix passing through the point with $\theta = \frac{1}{2}\pi$ and $\varphi = 0.3225$, and the region below the separatrix corresponds to rotations. The critical points are $\left(\frac{1}{2}\pi, 0\right)$, $\left(\frac{1}{2}\pi, \frac{1}{2}\pi\right)$, $\left(0.6847, \frac{1}{2}\pi\right)$, and $(0.8861, 0)$. For case g_2 ($\mu = -2$, $\alpha = -5$) the phase curves demonstrate a similar behavior; in this case the separatrix passes through the point with $\theta = \frac{1}{2}\pi$ and $\varphi = 0.9553$, and the critical points are $\left(\frac{1}{2}\pi, 0\right)$, $\left(\frac{1}{2}\pi, \frac{1}{2}\pi\right)$, $\left(0.8861, \frac{1}{2}\pi\right)$, and $(0.6847, 0)$.

5. SPECIAL CASES OF MOTION OF THE BODY

The value $\theta = 0$ is a critical point for the first equation in system (4.2). The equation for φ in the case $\theta = 0$ becomes a separable equation. The integration of this equation yields

$$\tan\varphi = l\tan[\pm r\xi + \arctan(l^{-1}\tan\varphi_0)], \qquad l = \sqrt{\mu/(\mu-1)}, \quad r = \sqrt{\mu(\mu-1)}, \quad \xi = \bar{L}_0\beta t, \tag{5.1}$$

where the upper and the lower signs of the variable r correspond to $\mu > 1$ and $\mu < 0$, respectively.

For $0 < \mu < 1$, we have

$$\tan\varphi = j\frac{a\exp(J\xi) - w}{a\exp(J\xi) + w}, \tag{5.2}$$

where

$$a = 1 + \sqrt{(1-\mu)/\mu}\,\tan\varphi_0, \quad J = 2\sqrt{\mu(1-\mu)}, \quad w = 1 - \sqrt{(1-\mu)/\mu}\,\tan\varphi_0, \quad j = \sqrt{\mu/(1-\mu)}.$$

For small θ, system (4.1) can be represented as

$$\theta' = \theta\sin\theta\cos\varphi, \quad \varphi' = \mu - \sin^2\varphi. \tag{5.3}$$

The terms of order > 1 in θ are omitted in system (5.3). For small θ, the equation for φ coincides with the corresponding equation at $\theta = 0$, and hence, its solution can be represented in the form (5.1) or (5.2).

By substituting (5.1) into the equation for θ in system (5.3) and by integrating this equation, we obtain

$$\theta^2 = \theta_0^2 l^{\mp 2}(l^2\cos^2\varphi_0 + \sin^2\varphi_0)^{\pm 1}\left\{\cos^2[\pm r\xi + \arctan(l^{-1}\tan\varphi_0)] + l^2\sin^2[\pm r\xi + \arctan(l^{-1}\tan\varphi_0)]\right\}^{\pm 1}, \tag{5.4}$$

where the upper and the lower signs correspond to $\mu > 1$ and $\mu < 0$, respectively.

For $0 < \mu < 1$, with allowance for (5.2), we obtain

$$\theta = \theta_0\sqrt{G\exp(2J\xi) + H\exp(J\xi) + V}\,\exp\left(-\tfrac{1}{2}J\xi\right), \tag{5.5}$$

where

$$G = a^2(1 + j^2), \quad H = 2aw(1 - j^2), \quad V = w^2(1 + j^2).$$

Thus, we have investigated the evolution of rotations of a nearly spherically symmetric satellite acted upon by the light pressure torque. The coefficient of the light pressure torque is approximated by a trigonometric polynomial of an arbitrary order. New qualitative properties of the rotations of the satellite are established.

ACKNOWLEDGMENT

The author thanks F. L. Chernousko, L. D. Akulenko, and N. N. Bolotnik for valuable advice, useful discussions, and attention to this work.

The work was partially financially supported by International Soros Science and Education Program in Ukraine (grant No. APU051058).

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