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Evolution of the Satellite Fast Rotation Due to the Gravitational Torque in a Dragging Medium

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Abstract—We study the fast rotational motion of a dynamically nonsymmetric satellite about the center of mass under the action of the gravitational torque and the drag torque. Orbital motions with arbitrary eccentricity are assumed to be given. The drag torque is assumed to be a linear function of the angular velocity. The system obtained after the averaging over the Euler–Poinsot motion is studied. We discover the following phenomena: the modulus of the angular momentum and the kinetic energy decrease, and there exist quasistationary regimes of motion (along the polhodes). The orientation of the angular momentum vector in the orbital frame of reference is determined. The general case is studied numerically, and an analytic study is performed in a neighborhood of the axial rotation and in the case of small dissipation.

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1. STATEMENT OF THE PROBLEM

Consider the motion of a satellite about the center of mass under the combined action of gravitational and drag torques. The rotational motions are considered in the framework of the model of dynamics of a rigid body whose center of mass moves in an elliptic orbit around the Earth. At present, dynamic problems generalized and complicated by taking into account various perturbation factors still remain topical. The papers [1–8] study rotational motions of bodies about a fixed point under the action of perturbing torques of various nature (gravitational, aerodynamic, electromagnetic, etc.); these studies are close to those in the present paper.

We introduce three Cartesian coordinate systems whose origins coincide with the satellite center of inertia [1, 2]. The coordinate system Ox_i $(i = 1, 2, 3)$ moves translationally together with the center of inertia: the axis Ox_1 is parallel to the position vector of the orbit perigee, the axis Ox_2 is parallel to the velocity vector of the satellite center of mass at the perigee, and the axis Ox_3 is parallel to the normal to the orbit plane. The coordinate system Oy_i $(i = 1, 2, 3)$ is attached to the satellite and oriented along the angular momentum vector $\mathbf{G}$. The axis Oy_3 is directed along $\mathbf{G}$, the axis Oy_2 lies in the orbit plane Ox_1x_2, and the axis Oy_1 lies in the plane Ox_3y_3 and is directed so that the vectors $\mathbf{y}_1$, $\mathbf{y}_2$, $\mathbf{y}_3$ form a right trihedral [1–3]. The axes of the coordinate system Oz_i $(i = 1, 2, 3)$ are related to the principal central axes of inertia of the rigid body. The mutual position of the principal central axes of inertia and the axes Oy_i is determined by the Euler angles. The direction cosines α_{ij} of the axes z_i with respect to the system Oy_i are expressed via the Euler angles φ, ψ, θ by well-known formulas [1]. The position of the angular momentum vector $\mathbf{G}$ with respect to the center of mass in the coordinate system Ox_i is determined by the angles λ and δ as shown in [1–3].

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The equations of motion of the body about the center of mass can be written in the form [2]

$$\begin{aligned}
&\frac{dG}{dt} = L_3, \quad \frac{d\delta}{dt} = \frac{L_1}{G}, \quad \frac{d\lambda}{dt} = \frac{L_2}{G\sin\delta},\\
&\frac{d\theta}{dt} = G\sin\theta\sin\varphi\cos\varphi\Big(\frac{1}{A_1} - \frac{1}{A_2}\Big) + \frac{L_2\cos\psi - L_1\sin\psi}{G},\\
&\frac{d\varphi}{dt} = G\cos\theta\Big(\frac{1}{A_3} - \frac{\sin^2\varphi}{A_1} - \frac{\cos^2\varphi}{A_2}\Big) + \frac{L_1\cos\psi + L_2\sin\psi}{G\sin\theta},\\
&\frac{d\psi}{dt} = G\Big(\frac{\sin^2\varphi}{A_1} + \frac{\cos^2\varphi}{A_2}\Big) - \frac{L_1\cos\psi + L_2\sin\psi}{G}\cot\theta - \frac{L_2}{G}\tan\delta.
\end{aligned} \tag{1.1}$$

Here the L_i are the external torques about the axes Oy_i, G is the value of the angular momentum, and the A_i ($i = 1, 2, 3$) are the principal central moments of inertia about the axes Oz_i.

We write the projections L_i of the external torques, formed by the gravitational torque L_i^g and the external drag torque L_i^τ, onto the axes Oy_i in the form introduced in [2, 6]. Here we present the projection onto the axis Oy_1 (the projections on the other axes can be written in a similar way):

$$\begin{aligned}
&L_1 = L_1^g + L_1^\tau \equiv \frac{3\omega_0^2(1+e\cos\theta)^3}{(1-e^2)^3}\sum_{j=1}^{3}(\beta_2\beta_j S_{3j} - \beta_3\beta_j S_{2j}) - G\sum_{i=1}^{3}\Big(\frac{I_{i1}\alpha_{1i}\alpha_{31}}{A_1} + \frac{I_{i2}\alpha_{1i}\alpha_{32}}{A_2} + \frac{I_{i3}\alpha_{1i}\alpha_{33}}{A_3}\Big),\\
&S_{mj} = \sum_{p=1}^{3} A_p\alpha_{jp}\alpha_{mp}, \quad \beta_1 = \cos(\nu-\lambda)\cos\delta, \quad \beta_2 = \sin(\nu-\lambda), \quad \beta_3 = \cos(\nu-\lambda)\sin\delta,
\end{aligned} \tag{1.2}$$

where ω_0 is the angular velocity of the orbital motion and e is the orbit eccentricity.

In some cases, along with the variable θ, it is convenient to use an important characteristic, the kinetic energy T, as an additional variable. Its derivative has the form

$$\begin{aligned}
\frac{dT}{dt} = \frac{2T}{G}L_3 + G\sin\theta\Big[&\cos\theta\Big(\frac{\sin^2\varphi}{A_1} + \frac{\cos^2\varphi}{A_2} - \frac{1}{A_3}\Big)(L_2\cos\psi - L_1\sin\psi)\\
&+ \sin\varphi\cos\varphi\Big(\frac{1}{A_1} - \frac{1}{A_2}\Big)(L_1\cos\psi + L_2\sin\psi)\Big].
\end{aligned} \tag{1.3}$$

The satellite center of mass moves along the Kepler ellipse with eccentricity e and revolution period Q. The true anomaly ν depends on time t as follows:

$$\frac{d\nu}{dt} = \frac{\omega_0(1+e\cos\nu)^2}{(1-e^2)^{3/2}}, \quad \omega_0 = \frac{2\pi}{Q}. \tag{1.4}$$

Consider a dynamically nonsymmetric satellite. To be definite, let its moments of inertia satisfy the inequalities $A_1 > A_2 > A_3$. We assume ~~that~~ that the angular velocity ω of the satellite motion about the center of mass is significantly larger than the angular velocity ω_0 of the orbital motion; i.e., $\varepsilon = \omega_0/\omega \sim A_1\omega_0/G \ll 1$. Then the kinetic energy of the body rotation is large compared with the perturbing torques.

In the present paper, we assume that the drag torque $\mathbf{L}^\tau$ can be represented as $\mathbf{L}^\tau = \mathbf{I}\boldsymbol{\omega}$, where the tensor $\mathbf{I}$ has constant components I_{ij} in the body-fixed frame Oz_i [1, 6]. We assume that the medium drag is weak and has the order of ε^2, $\|\mathbf{I}\|/G_0 \sim \varepsilon^2 \ll 1$, where $\|\mathbf{I}\|$ is the norm of the matrix of the drag coefficients and G_0 is the satellite angular momentum at the initial time.

We pose the problem of studying the solution of system (1.1)–(1.4) for small ε on a large time interval $t \sim \varepsilon^{-2}$. We solve this problem by the averaging method [9].

2. THE AVERAGING METHOD PROCEDURE

Consider the unperturbed motion ($\varepsilon = 0$), where the external torques are zero. Then the rotation of the rigid body is an Euler–Poinsot motion. The variables G, δ, λ, T, and ν become constants, and φ, ψ, and θ are functions of time t. The slow variables in the perturbed motion are G, δ, λ, T, and ν, and the fast variables are the Euler angles φ, ψ, and θ.

Consider the motion under the condition $2TA_1 \geq G^2 \geq 2TA_2$ corresponding to case in which the trajectories of the angular momentum vector surround the axis Oz_1 of the maximal moment of inertia [10]. We introduce the quantity

$$k^2 = \frac{(A_2 - A_3)(2TA_1 - G^2)}{(A_1 - A_2)(G^2 - 2TA_3)}, \quad 0 \leq k^2 \leq 1. \tag{2.1}$$

In the unperturbed motion, it is a constant, namely, the modulus of the elliptic functions describing this motion.

To construct the averaged system in the first approximation, we substitute the solution of the unperturbed Euler–Poinsot motion [10] into the right-hand sides of Eqs. (1.1) and (1.3) and perform averaging over the variable ψ and then over the time t with the dependencies of φ and θ on t [2] taken into account. The previous notation for the slow variables δ, λ, G, and T is preserved. As a result, we obtain the following four equations:

$$\frac{d\delta}{dt} = -\frac{3\omega_0^2(1 + e\cos\nu)^3}{2G(1 - e^2)^3}\beta_2\beta_3 N^*, \quad \frac{d\lambda}{dt} = \frac{3\omega_0^2(1 + e\cos\nu)^3}{2G(1 - e^2)^3 \sin\delta}\beta_1\beta_3 N^*,$$

$$\frac{dG}{dt} = -\frac{G}{R(k)}\{I_{22}(A_1 - A_3)W(k) + I_{33}(A_1 - A_2)[k^2 - W(k)] + I_{11}(A_2 - A_3)[1 - W(k)]\},$$

$$\begin{aligned} \frac{dT}{dt} = -\frac{2T}{R(k)}\Big\{ & I_{22}(A_1 - A_3)W(k) + I_{33}(A_1 - A_2)[k^2 - W(k)] \\ & + \frac{(A_1 - A_2)(A_1 - A_3)(A_2 - A_3)}{S(k)}\Big[\frac{I_{33}}{A_3}[k^2 - W(k)] + \frac{I_{22}}{A_2}(1 - k^2)W(k)\Big] \\ & + \frac{I_{11}}{A_1}\frac{(A_2 - A_3)R(k)}{S(k)}[1 - W(k)]\Big\}, \end{aligned} \tag{2.2}$$

$$W(k) = 1 - \frac{E(k)}{K(k)}, \quad R(k) = A_1(A_2 - A_3) + A_3(A_1 - A_2)k^2, \quad S(k) = A_2 - A_3 + (A_1 - A_2)k^2,$$

$$N^* = A_2 + A_3 - 2A_1 + 3\Big(\frac{2A_1T}{G^2} - 1\Big)\Big[A_3 + (A_2 - A_3)\frac{K(k) - E(k)}{K(k)k^2}\Big].$$

Here $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind, respectively. By differentiating the expression (2.1) for k^2 and by using the last two equations in (2.2), we obtain a differential equation independent of the other variables:

$$\begin{aligned} & \frac{dk^2}{d\xi} = (1 - \chi)(1 - k^2) - [(1 - \chi) + (1 + \chi)k^2]\frac{E(k)}{K(k)}, \\ & \chi = \frac{2I_{22}A_1A_3 - I_{11}A_2A_3 - I_{33}A_1A_2}{(I_{33}A_1 - I_{11}A_3)A_2}, \quad \xi = \frac{t - t_*}{N}, \quad N = \frac{A_1A_3}{I_{33}A_1 - I_{11}A_3} \sim \varepsilon^{-2}. \end{aligned} \tag{2.3}$$

Here t_* is a constant. The value $k^2 = 1$ is associated with the relation $2TA_2 = G^2$, which corresponds to the separatrix for the Euler–Poinsot motion.

It follows from Eqs. (2.2) that the medium drag results in the evolution of both the body kinetic energy T and the absolute value G of the angular momentum. It is easy to see that in the first approximation only the drag force causes them to vary and the equations contain only the diagonal coefficients I_{ii} of the friction torque matrix. The terms containing the off-diagonal components I_{ij} $(i \neq j)$ disappear after the averaging. The variations in the angles λ and δ are caused both by the drag force and the gravitational attraction.

Equation (2.3) describes the averaged motion of the endpoint of the angular momentum vector **G** on the sphere of radius G. The third equation in (2.2) describes the variation in the sphere radius in the course of time.

The expression in braces on the right-hand side of the equation for G in (2.2) is positive (for $A_1 > A_2 > A_3$), because the inequalities $(1 - k^2)K \leq E \leq K$ are satisfied [11]. The coefficient of each I_{ii} is a negative function of k^2, and moreover, all of them cannot be zero simultaneously.

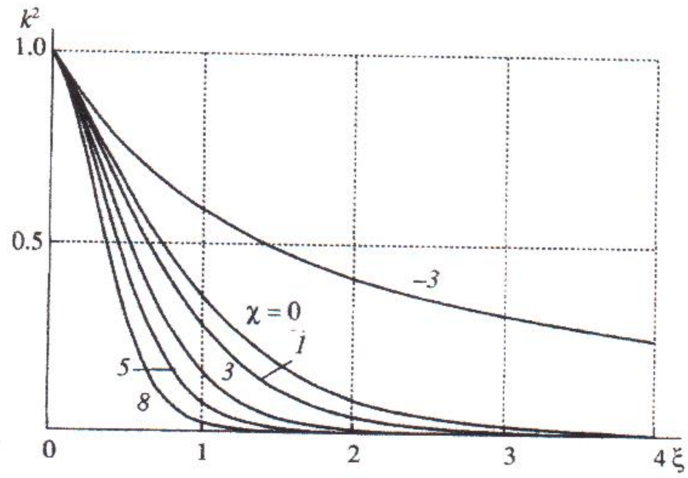

Fig. 1.

Since $G > 0$, we have $dG/dt < 0$; i.e., the variable G strictly decreases for any $k^2 \in [0, 1]$. In a similar way, one can show that the kinetic energy also strictly decreases.

The main stage in the study of the body motion is the analysis of Eq. (2.3). Note that the evolution of k^2 is affected only by the medium drag, and since this equation can be integrated independently, the influence of the gravitational moment and the drag torque is partly separated. The total separation is impossible in this case, because the slowly decreasing variables G and T occur on the right-hand sides of the equations for λ and δ in (2.2). Equation (2.3) coincides with the similar equation for the free spatial motion of a body with a cavity filled with a fluid of large viscosity [12] and with the equation describing the motion of a heavy rigid body in a dragging medium [6].

One can readily verify that the variable χ in (2.3) satisfies the relation

$$\chi = \frac{A_3\chi_1 - A_1\chi_2}{A_3\chi_1 + A_1\chi_2}, \quad \chi_1 = I_{22}A_1 - I_{11}A_2, \quad \chi_2 = I_{33}A_2 - I_{22}A_3,$$

which implies that, since the variables χ_1 and χ_2 can take any values, the variable χ varies in the range $(-\infty, +\infty)$ depending on the problem parameters A_i and I_{ii} $(i = 1, 2, 3)$. The case in which inequalities $\chi_1 > 0$ and $\chi_2 > 0$ hold and hence $|\chi| \leq 1$ was studied in [12]. An equation of the form (2.3) for a rigid body with a cavity filled with a strongly viscous fluid, where the parameter χ varies in the range $|\chi| \leq 3$, was considered in [13]. A similar equation holds for $\chi \in (-\infty, +\infty)$ in the case of a fast motion of a heavy body about a fixed point in a dragging medium [6].

The numerical integration of Eq. (2.3) with the initial condition $k^2(0) \approx 1$ shows that the function k^2 decreases monotonically as ξ increases, and the larger χ, the faster this decrease is. Numerical calculations performed for Eq. (2.3) are shown in Fig. 1 for $\chi = -3; 0; 1; 3; 5; 8$. One can see that the larger χ, the faster the function k^2 decreases. We note that some new qualitative effects appear for $\chi < -3$, and for $\chi > 3$, the character of the solution remains the same as for $|\chi| \leq 3$.

Equation (2.3) for k^2 admits stationary points $k^2 = k_*^2$ for $\chi < -3$, where, independently of G and T, the variable k^2 remains constant by virtue of Eq. (2.3) for an appropriate choice of the initial conditions. We note that for $\chi > -3$ such stationary points (except for $k = 0$ and $k = 1$) do not exist.

To find the quasistationary solutions $k^2 = k_*^2$, we equate the right-hand side of (2.3) with zero and solve the equation thus obtained for χ:

$$\chi = \frac{k^2 - 1 + (1 + k^2)E(k)/K(k)}{(1 - k^2)[E(k)/K(k) - 1]}. \tag{2.4}$$

The graph of the dependence of χ on k^2, determined numerically, is presented by curve *1* in Fig. 2, which shows that for any $\chi < -3$ there exists a unique value $k_*^2 \in (0, 1)$ corresponding to the quasistationary motion. The numerical calculations were performed for $k_*^2 = 0.2; 0.4; 0.6; 0.8$. In Fig. 3,

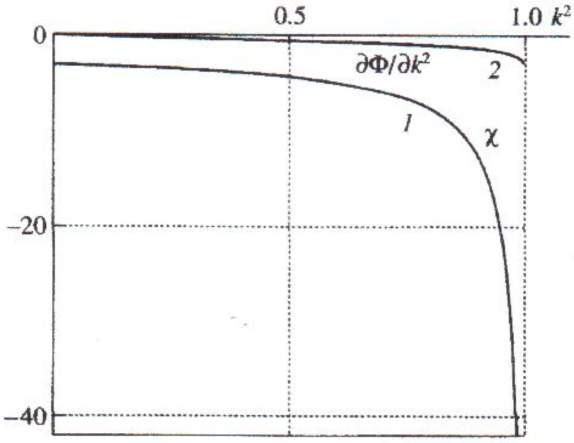

Fig. 2.

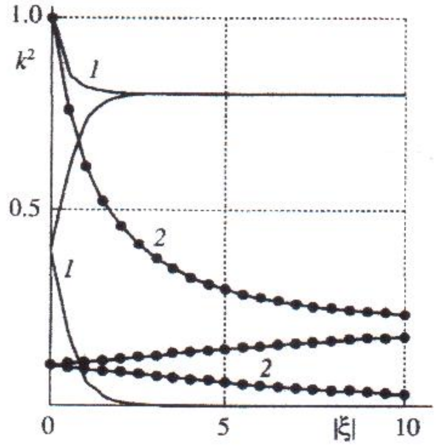

Fig. 3.

we see typical graphs of the functions $k^2(\chi, \xi)$ obtained by the numerical integration of Eq. (2.3). The solid curve was obtained for $k_*^2 = 0.8$, and the curve with filled symbols was obtained for $k_*^2 = 0.2$. For the given values of k_*^2, the value $\chi = \chi_*$ was determined according to Eq. (2.4). Then Eq. (2.3) was integrated numerically for the value of χ_* thus obtained. Each graph contains three branches. The initial condition for the upper branches was taken in the form $k^2(0) = 1$. The lower two branches for each graph were constructed for the initial conditions $k^2(0) = k_*^2/2$. In this case, the increasing branch corresponds to integration for $\xi > 0$, and the decreasing branch is the mirror reflection in the straight line $\xi = 0$ of the dependence $k^2(\chi, \xi)$ obtained for $\xi < 0$.

Equation (2.3) is autonomous; therefore, the solution $k^2(\chi, \xi)$ can be determined for arbitrary initial conditions. The choice of the corresponding branch of the graph permits determining the character of the variation in k^2. The upper branch is taken for the initial value $k^2 = k_0^2 > k_*^2$; if $k_*^2/2 \leq k_0^2 < k_*^2$, then the middle branch is taken. If $k_0^2 \leq k_*^2/2$, then the lower branch is taken, for which the motion occurs with negative ξ as k^2 increases until $k^2 = k_*^2/2$; after that, we switch to the middle branch.

Consider the system consisting of the first two equations in system (2.2) and Eq. (1.4). They can be written as

$$\dot{\delta} = \omega_0^2 \Delta(\nu, \delta, \lambda), \quad \dot{\lambda} = \omega_0^2 \Lambda(\nu, \delta, \lambda), \quad \dot{\nu} = \frac{\omega_0}{h(e)}(1 + e\cos\nu)^2, \quad h(e) = (1 - e^2)^{3/2}.$$

Here Δ and Λ are the coefficients on the right-hand sides in the first two equations in (2.2), δ and λ are slow variables, and ν is a semislow variable.

We obtain a system of special form, which we solve by a modified averaging method according to the following scheme [14]:

$$\dot{\delta} = \frac{\omega_0^2 h(e)}{2\pi} \int_0^{2\pi} \frac{\Delta(\lambda, \delta, \nu)}{(1 + e\cos\nu)^2} d\nu, \quad \dot{\lambda} = \frac{\omega_0^2 h(e)}{2\pi} \int_0^{2\pi} \frac{\Lambda(\lambda, \delta, \nu)}{(1 + e\cos\nu)^2} d\nu.$$

After the averaging, we have

$$\dot{\delta} = 0, \quad \dot{\lambda} = \frac{3\omega_0^2 N^* \cos\delta}{4Gh(e)}. \tag{2.5}$$

We preserve the notation for the slow averaged variables. We note that the action of the applied forces does not change the angular velocity δ and that the deviation of the vector $\mathbf{G}$ from the vertical remains constant in this approximation.

The resulting system (2.5), the last two equations in system (2.2), and the equation

$$\frac{dk^2}{dt} = \frac{I_{33}A_1 - I_{11}A_3}{A_1 A_3}\left\{(1-\chi)(1-k^2) - [(1-\chi) + (1+\chi)k^2]\frac{E(k)}{K(k)}\right\} \tag{2.6}$$

can be integrated numerically. All these equations are nondimensionalized and are considered for small time $\tau \sim \varepsilon^2 t$. The equations for the derivatives of the kinetic energy T and the angular momentum G in system (2.2), as well as the dimensionless equation for k^2 of the form (2.6) with respect to the small time τ, are written in the same form. System (2.5) becomes

$$\dot{\delta} = 0, \quad \dot{\lambda} = \frac{3N^* \cos\delta}{4Gh(e)}.$$

The integration was carried out for the initial conditions $G(0) = 1$, $k^2(0) = 0.99$, $\delta(0) = 0.785$ rad, and $\lambda(0) = 0.785$ rad and for the following values of the principal central moments of inertia of the body: $A_1 = 3.2$, $A_2 = 2.6$, and $A_3 = 1.67$. Numerical calculations were performed for different types of orbits with eccentricity $e = 0$ for the circular orbit, $e = 0.04473$ for the orbit of the first Soviet satellite, $e = 0.0487$ for the orbit of the third Soviet satellite, and $e = 0.421$ for a strongly elliptic orbit [1]. Two possible versions of the drag coefficients were considered: $I_{11} = 2.322$, $I_{22} = 1.31$, $I_{33} = 1.425$ and $I_{11} = 0.919$, $I_{22} = 5.288$, $I_{33} = 1.666$. In the first case, χ in Eq. (2.6) was negative, namely, $\chi = -4.477$; in the second case, $\chi = 3.853$. Numerical analysis shows that the functions $G(t)$ and $T(t)$ are monotone decreasing (Figs. 4 and 5). One can see that for positive χ (curves *2*) the functions decrease faster but the function $G(t)$ tends to the asymptote slower for a larger time interval. In both computational versions, the function $\lambda = \lambda(t)$ of the variable χ is a decreasing function, but it decreases faster in the first version. It should be noted that in both versions an increase in the orbit eccentricity e results in a faster decrease of the angle λ. In Fig. 6, we present the graphs of the function $\lambda = \lambda(t)$ for $e = 0$ (curve *1*) and $e = 0.421$ (curve *2*) for positive χ. One can see that the angle λ decreases with time; i.e., the vector $\mathbf{G}$ rotates clockwise around the normal to the orbit plane and remains at a constant angular distance δ from it.

3. ANALYSIS OF THE LIMIT CASES

3.1. Consider the body motion for small $k^2 \ll 1$, which corresponds to the motions of a rigid body close to rotations about the axis A_1. In this case, one can simplify the right-hand side of Eq. (2.3) by using the expansions of complete elliptic integrals in series in k^2 [11]. Then (2.3) can be integrated, and the asymptotic solution has the form

$$k^2 = C_1 \exp\left[-\frac{(3+\chi)\xi}{2}\right] = k_0^2 \exp(-\rho t), \quad \rho = \alpha_2 + \alpha_3 - 2\alpha_1, \quad \alpha_i = \frac{I_{ii}}{A_i} \quad (i = 1, 2, 3), \tag{3.1}$$

where $C_1 > 0$ is a constant.

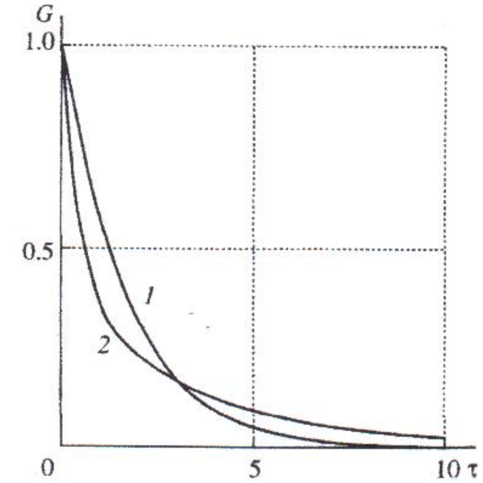

Fig. 4.

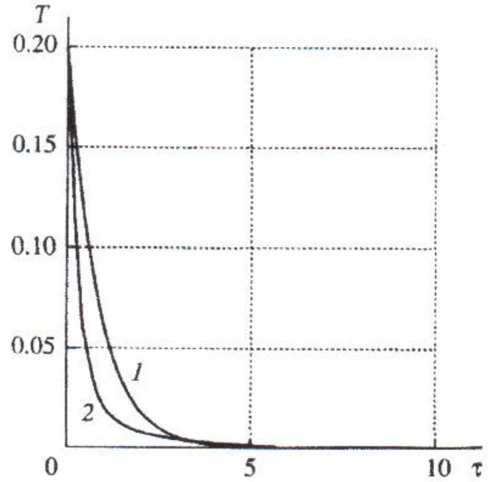

Fig. 5.

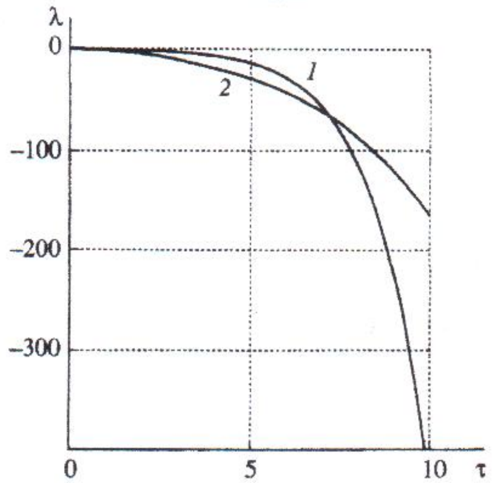

Fig. 6.

For small k^2, the analytic expressions for the angular momentum and the kinetic energy can be obtained explicitly:

$$G = G_0 \exp[-\alpha_1 t + b\exp(-\rho t)], \quad T = T_0 \exp[-2\alpha_1 t + a\exp(-\rho t)],$$

$$b = \frac{k_0^2}{2\rho A_1^2(A_2 - A_3)}[\alpha_1 A_1(2A_2A_3 - A_1A_2 - A_1A_3) + \alpha_2 A_1 A_2(A_1 - A_3) + \alpha_3 A_1 A_3(A_1 - A_2)], \qquad (3.2)$$

$$a = \frac{k_0^2}{(A_2 - A_3)\rho}[\alpha_2(A_1 - A_3) + \alpha_3(A_1 - A_2) + \alpha_1(A_2 + A_3 - 2A_1)].$$

Equation (2.5) for λ with (3.2) taken into account can be written as

$$\frac{d\lambda}{dt} = \phi\exp[\alpha_1 t - b\exp(-\rho t)]\big(d + \mu\{\eta\exp[(a - 2b)\exp(-\rho t)] - 1\}\big),$$

$$\phi = \frac{3\omega_0^2\cos\delta}{4(1 - e^2)^{3/2}G_0}, \quad d = A_2 + A_3 - 2A_1, \quad \mu = \frac{3}{2}(A_2 + A_3), \quad \eta = \frac{2A_1T_0}{G_0}.$$

Its solution has the form

$$\lambda = \frac{\phi}{\rho}\{(\mu - d)b^k[-\gamma(-k, b) + \gamma(-k, be^\tau)] - \mu\eta x^k[-\gamma(-k, x) + \gamma(-k, xe^\tau)]\},$$

$$k = \frac{\alpha_1}{\rho}, \quad x = 3b - a, \quad \tau = -\rho t.$$

Here $\gamma(n, x)$ is the incomplete gamma function [11], $b > 0$, and $x > 0$.

3.2. It is of interest to study system (2.2) in the case of small diagonal drag coefficients, i.e., for

$$I_{11} = \mu i_{11}, \quad I_{22} = \mu i_{22}, \quad I_{33} = \mu i_{33}, \quad \mu \ll 1. \qquad (3.3)$$

The angular momentum function G and the kinetic energy function T can be represented as power series in μ:

$$G = G_0 + \mu G_1 + \dots, \quad T = T_0 + \mu T_1 + \dots$$

After integration, the last two equations in (2.2) can be written as

$$G = G_0 - \frac{G_0\mu t}{R(k_0)}\{i_{22}(A_1 - A_3)W(k_0) + i_{33}(A_1 - A_2)[k_0^2 - W(k_0)] + i_{11}(A_2 - A_3)[1 - W(k_0)]\},$$

$$T = T_0 - \frac{2T_0\mu t}{R(k_0)}\Big\{i_{22}(A_1 - A_3)W(k_0) + i_{33}(A_1 - A_2)[k_0^2 - W(k_0)]$$
$$+ \frac{(A_1 - A_2)(A_1 - A_3)(A_2 - A_3)}{S(k_0)}\Big[\frac{i_{33}}{A_3}[k_0^2 - W(k_0)] + \frac{i_{22}}{A_2}(1 - k_0^2)W(k_0)\Big]$$
$$+ \frac{i_{11}}{A_1}\frac{(A_2 - A_3)R(k_0)}{S(k_0)}[1 - W(k_0)]\Big\}. \qquad (3.4)$$

Here $W(k_0)$, $R(k_0)$, and $S(k_0)$ are the values of the functions (2.2) for $k = k_0$. According to (3.4), the functions $G(t)$ and $T(t)$ are strictly decreasing, just as in the case of system (2.2).

For small drag torques, it is necessary to construct the approximate solution

$$k^2 = k_0^2 + \frac{2\mu t}{A_1A_2A_3}\Big\{A_1(i_{33}A_2 - i_{22}A_3)(1 - k_0^2) - [A_1(i_{33}A_2 - i_{22}A_3) + A_3(i_{22}A_1 - i_{11}A_2)k_0^2]\frac{E(k_0)}{K(k_0)}\Big\}.$$

We use formula (3.4) for the variation in the angular momentum to analyze the direction of the vector **G**. According to (2.5), the deviation in the vector **G** from the vertical also remains constant, just as in the case of small k^2, and the rate of variation in the angle λ depends on the nonconstant variable N^*, which can be expressed via the angular momentum G and the kinetic energy T. Then, for small drag torques, the law of time variations in the angle λ has the form

$$\lambda = \lambda_0 + \frac{3\omega_0^2 N_0 t\cos\delta}{4G_0(1 - e^2)^{3/2}} + \frac{3\omega_0^2\mu t^2\phi\cos\delta}{8G_0(1 - e^2)^{3/2}}.$$

The variation in the angle $\lambda = \lambda(t)$ has the form of a quadratic function of t whose constant term and the coefficient of the first power of t can be expressed via the constants ω_0, N_0, G_0, and $\cos\delta$. All these quantities are positive and are specified at the initial time moment.

3.3. It is of interest to consider the case of small k^2 and small drag coefficients (3.3). For small k^2, we obtain the laws of variations in the angular momentum G and the kinetic energy T (3.2), which, with the terms of the order of μ taken into account, give

$$\begin{gathered}
G = G_0\{1 + m - nt\}, \quad T = T_0\{1 + q - \zeta t\}, \\
n = \mu\left\{\alpha_{1\mu} + \frac{k_0^2}{2A_1^2(A_2 - A_3)}[\alpha_{1\mu}A_1(A_2A_3 - A_1A_2 - A_1A_3) + \alpha_{2\mu}A_1A_2(A_1 - A_3) + \alpha_{3\mu}A_1A_3(A_1 - A_2)]\right\}, \\
m = \frac{k_0^2}{2\rho_\mu A_1^2(A_2 - A_3)}[\alpha_{1\mu}A_1(A_2A_3 - A_1A_2 - A_1A_3) + \alpha_{2\mu}A_1A_2(A_1 - A_3) + \alpha_{3\mu}A_1A_3(A_1 - A_2)], \\
q = \frac{k_0^2}{(A_2 - A_3)\rho_\mu}[\alpha_{2\mu}(A_1 - A_3) + \alpha_{3\mu}(A_1 - A_2) + \alpha_{1\mu}(A_2 + A_3 - 2A_1)], \\
\zeta = \mu\left\{2\alpha_{1\mu} + \frac{k_0^2}{A_2 - A_3}[\alpha_{2\mu}(A_1 - A_3) + \alpha_{3\mu}(A_1 - A_2) + \alpha_{1\mu}(A_2 + A_3 - 2A_1)]\right\}, \\
\rho_\mu = \alpha_{2\mu} + \alpha_{3\mu} - 2\alpha_{1\mu}, \quad \alpha_{i\mu} = \frac{I_{ii}}{A_i} \quad (i = 1, 2, 3).
\end{gathered} \tag{3.5}$$

Just as in all preceding cases, the functions $G(t)$ and $T(t)$ are strictly decreasing. To determine the direction of rotation of the vector $\mathbf{G}$, we have reduced the second equation in system (2.2) to a different form with (3.5) taken into account. After integration, we obtain

$$\lambda = \frac{3\omega_0\cos\delta}{4(1 - e^2)^{3/2}G_0(1 + m)}\left((z + f)t + \left\{-\frac{nz}{1 + m} + f\left[\frac{3n(1 + q)}{1 + m} - \zeta\right]\right\}\frac{t^2}{2}\right) + \lambda_0,$$

$$z = -\frac{A_2 + A_3 + 4A_1}{2}, \quad f = 3(A_2 + A_3)\frac{T_0A_1(1 + q)}{G_0^2(1 + m)^2}.$$

4. QUALITATIVE PICTURE OF MOTION

Consider the stability of the quasistationary motions obtained in Secs. 2 and 3. We denote the right-hand side of Eq. (2.3) by $\Phi(k^2, \chi)$. The derivative of this function with respect to k^2 for χ corresponding to the quasistationary motions (2.4) has the form

$$\frac{\partial\Phi}{\partial k^2} = -\frac{2}{1 - k^2}\frac{E(k)}{K(k)} + \frac{1}{E(k)/K(k) - 1}\left\{2\frac{E(k)}{K(k)} - 1 - \frac{1}{1 - k^2}\left[\frac{E(k)}{K(k)}\right]^2\right\}.$$

In Fig. 2, curve *2* presents the graph of the function $\partial\Phi/\partial k^2$ obtained numerically. This graph shows that $\partial\Phi/\partial k^2 < 0$ for all $k^2 \in [0, 1]$; i.e., all quasistationary motions in Sec. 2 are asymptotically stable with respect to the variable k^2 (in the sense of [15]) for $\xi \geq 0$. This is also confirmed by the curves in Fig. 3.

The positive ξ are associated with true time $t \geq t_*$ and $N > 0$; i.e., $I_{33}A_1 > I_{11}A_3$ (2.3). However, these quasistationary motions are unstable in true time for $N < 0$ and $I_{33}A_1 < I_{11}A_3$.

In the case of small k^2, which corresponds to the motions of a rigid body close to rotations about the axis A_1, the functions $\Phi(k^2, \chi)$ and the derivative $\partial\Phi/\partial k^2$ have the form

$$\Phi(k^2, \chi) = -\frac{k^2}{2}(3 + \chi), \quad \frac{\partial\Phi}{\partial k^2} = -\frac{3 + \chi}{2}.$$

Thus, the quasistationary motion with $k^2 = 0$ and $\xi > 0$ is asymptotically stable if $\chi > -3$ and unstable if $\chi < -3$. In true time, this motion can be either stable or unstable for $t \geq t_*$ depending on the value of χ ($\chi > -3$ or $\chi < -3$) and the sign of the parameter N.

On the basis of the preceding, we obtain the following qualitative picture of motion. First, consider the case $N > 0$. For $t \geq t_*$, the motion is described by formulas (2.1)–(2.3) and the function (3.1) and

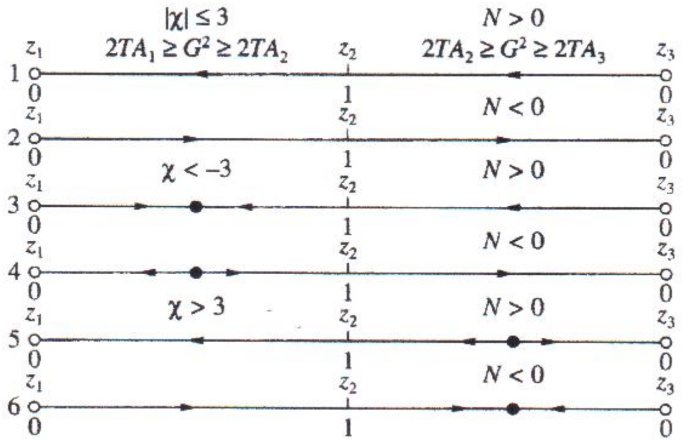

Fig. 7.

corresponds to the domain $2TA_1 \geq G^2 \geq 2TA_2$. For $t \leq t_*$, we consider the domain $2TA_2 \geq G^2 \geq 2TA_3$, which corresponds to the trajectories of the angular momentum vector surrounding the axis A_3. In this domain, we should interchange A_1 with A_3 and I_{11} with I_{33} in formulas (2.1)–(2.3). Equation (2.3) preserves its form but with χ replaced by $-\chi$ and N replaced by $-N$. In a similar way, we can analyze the motions for $N < 0$. We choose the constant t_* in formula (2.3) so that the motion at time $t = t_*$ corresponds to the passage through the separatrix.

Let us construct Fig. 7, which illustrates the character of variation in k^2 depending on χ and N in true time t. The points on the segments correspond to quasistationary motions. For $|\chi| \leq 3$, we have two points, $k^2 = 0$ and $k^2 = 1$. For $\chi > 3$ and $\chi < -3$, we have three points, $k^2 = 0$; k_*^2; 1. The arrows indicate the stability or instability of the quasistationary motion. The letters z_1, z_2, and z_3 denote the body axes corresponding to the indicated values of k^2. The left-hand side corresponds to the domain $2TA_1 \geq G^2 \geq 2TA_2$, and the right-hand side corresponds to the domain $2TA_2 \geq G^2 \geq 2TA_3$. Six possible combinations of χ and N are considered.

We interpret the results as follows. In formula (3.1), we have introduced the notation α_i ($i = 1, 2, 3$). These quantities have the meaning of damping coefficients for rotations about the principal axes of inertia Oz_i. For example, the rotation of the rigid body about the axis Oz_1 under the action of the dissipative torque proportional to the first power of the angular velocity ω is described by the relations

$$A_1 \frac{d\omega}{dt} = -I_{11}\omega, \quad \omega = \omega^0 \exp(-\alpha_1 t).$$

We introduce the dimensionless variables $\beta_i = \alpha_i/\alpha_2$ ($i = 1, 2, 3$) and rewrite relation (2.3) for χ and N as

$$\chi = \frac{2 - \beta_1 - \beta_3}{\beta_3 - \beta_1}, \quad N = \frac{1}{\alpha_2(\beta_3 - \beta_1)}. \tag{4.1}$$

On the straight line $\beta_3 = \beta_1$, the variable N changes its sign, and, by relation (4.1), the straight lines $\beta_3 = (1 + \beta_1)/2$ and $\beta_3 = 2\beta_1 + 1$ correspond to $\chi = \pm 3$. We draw these lines in the coordinate plane $(\beta_1\beta_3)$ for $\beta_1 > 0$ and $\beta_3 > 0$. The quadrant is divided into six domains (Fig. 8) numbered in accordance with the order numbers in Fig. 7. One can see that the number of quasistationary regimes of motion and their instability depend on the relative values α_i ($i = 1, 2, 3$) of rotation damping about the principal axes of inertia.

Thus, in the approximation under study, the perturbed motion of the body consists of a fast Euler–Poinsot motion about the vector $\mathbf{G}$ and a slow evolution of the parameters of this motion. The angular momentum and the kinetic energy decrease strictly, and their variation depends only on the medium drag torque. In the first approximation, the motion of the angular momentum vector $\mathbf{G}$ about the vertical on the orbit plane is described by the first two equations in system (2.2). The velocity of rotation of the vector $\mathbf{G}$ about the vertical varies, and so does the deviation of the vector from the vertical. In the

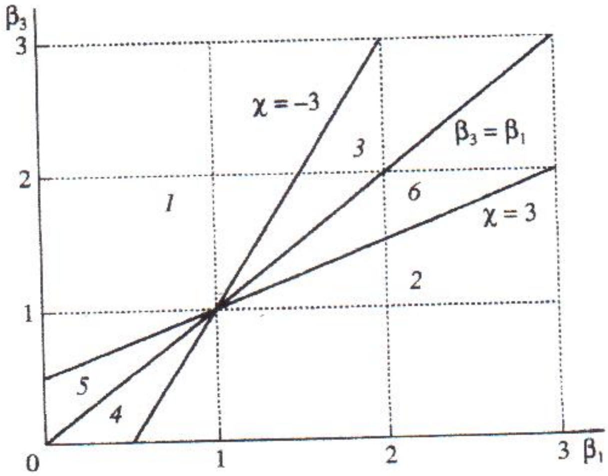

Fig. 8.

second approximation of the averaging method, the deviation of the vector **G** from the vertical remains constant, and the angular velocity of rotation in this case is variable, see (2.5). The evolution of the parameters of the Euler–Poinsot motion in the body-fixed coordinate system is described by Eq. (2.3) and is qualitatively represented in Fig. 7.

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