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RELATIVE OSCILLATIONS AND ROTATIONS IN A PLANE TWO-RIGID-BODY HINGED SYSTEM

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The problem of the free motions of a hinged two-body system was discussed in [1]. The equations were integrated completely on the basis of two system integrals, the general solution was obtained in implicit form, and the motions were investigated qualitatively and numerically. In practical cases, however, the system is acted upon by various perturbations and controlling forces and moments of forces, consideration of which requires analytic representation of the general and particular solutions. In this paper, analytic methods of Lyapunov-Poincare nonlinear mechanics are used to construct approximate explicit solutions of the oscillatory and rotational types and recurrent procedures are proposed for their improvement. It is established that the motions are generally two-frequency (quasi-periodic) and conditions of path periodicity are obtained. The model considered here is of practical interest for solution of problems in the dynamics and control of complex engineered objects, industrial robots, space vehicles, etc.

1. STATEMENT OF THE PROBLEM

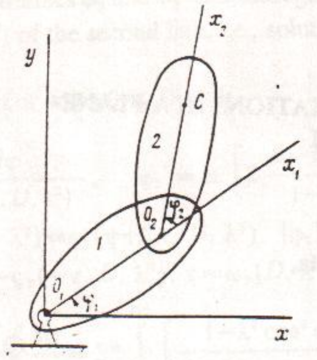
We shall consider free motions (with no external forces or moments of forces) of a plane system of two rigid bodies (links) in the inertial O_1xy plane (see the figure). The O_1z_1 axis is orthogonal to the plane of the figure and nonmoving in inertial space. The O_2z_2 axis is collinear with the O_1z_1 axis; it is nonmoving in bodies 1 and 2 and is the axis of the connecting cylindrical hinge, which, like O_1z_1 is assumed to be ideal. We introduce notation: $|O_1O_2| = l_1$ is the distance between the hinge axes, $|O_2C| = l_2$ is the "arm" of the second body about the O_2z_2 axis (the distance from point O_2 to the center of mass C of body 2); J_1 and J_2 are the moments of inertia of links 1 and 2 about axes O_1z_1 and O_2z_2, respectively; m_2 is the mass of body 2; the mass m_1 of the first body is immaterial to the analysis.

The angle variables ϕ_1 and ϕ_2 as indicated on the figure are convenient generalized variables that describe the motion of the system. Here ϕ_1 is the rotation angle of segment O_1O_2 (or of the O_1x_1 axis) about the O_1x axis and ϕ_2 is the angle that determines the rotation of segment O_2C about the continuation of segment O_1O_2. The kinetic (and total) energy E of the system is constant in the present case of free motion and can be written as a strictly positive quadratic form of $\phi_1^{\cdot}$, $\phi_2^{\cdot}$:

$$E={}^1/_2(J_*+2\mu\cos\varphi_2)\varphi_1^{\cdot 2}+{}^1/_2J_2\varphi_2^{\cdot 2}+(J_2+\mu\cos\varphi_2)\varphi_1^{\cdot}\varphi_2^{\cdot}$$
$$E=e=\text{const}\ (J_*=J_1+J_2+m_2l_1^2,\ \mu=m_2l_1l_2<{}^1/_2J_*) \tag{1.1}$$

The equations of motion can be written in the Newton, Lagrange, Routh, or Hamilton form [2]; for the equations in the Lagrange form, for example, we obtain the expressions

$$(J_*+2\mu\cos\varphi_2)\varphi_1^{\cdot\cdot}+(J_2+\mu\cos\varphi_2)\varphi_2^{\cdot\cdot}-\mu(\varphi_1^{\cdot}+2\varphi_2^{\cdot})\varphi_2^{\cdot}\sin\varphi_2=0$$
$$J_2\varphi_2^{\cdot\cdot}+(J_2+\mu\cos\varphi_2)\varphi_1^{\cdot\cdot}+\mu\varphi_1^{\cdot 2}\sin\varphi_2=0$$
$$\partial E/\partial\varphi_1^{\cdot}\equiv K_1=(J_*+2\mu\cos\varphi_2)\varphi_1^{\cdot}+(J_2+\mu\cos\varphi_2)\varphi_2^{\cdot}=k=\text{const} \tag{1.2}$$
$$\partial E/\partial\varphi_2^{\cdot}\equiv K_2=J_2\varphi_2^{\cdot}+(J_2+\mu\cos\varphi_2)\varphi_1^{\cdot}$$
$$(K_1^{\cdot}=0,\ K_2^{\cdot}=\partial E/\partial\varphi_2)$$



For $\mu = 0$, according to (1.2), the angular momentum $K_2 = k_2 = \text{const}$; the result is that $\phi_1^{\cdot} = \text{const}$, $\phi_2^{\cdot} = \text{const}$.

We note that when $J_* \gg J_2$ $(J_2/J_* \to 0)$ it follows from the first equation of (1.2) that $\phi_1^{\cdot} = \text{const}$, while the second equation describes the oscillations and rotations of a "pendulum-type" system: $\varphi_2^{\cdot\cdot} - a^2 \sin \varphi_2 = 0$, where $a^2 = \mu \varphi_1^{\cdot 2} / J_2 = \text{const} > 0$.

From (1.2) we evaluate $\varphi_1^{\cdot} = (k - B\varphi_2^{\cdot}) A$, where the coefficients $A = J_* + 2\mu \cos \varphi_2 > 0, B = J_2 + \mu \cos \varphi_2 \geqslant 0$ are multipliers before $\phi_1^{\cdot}$, $\phi_2^{\cdot}$, respectively. Substituting $\phi_1^{\cdot}$ into integral (1.1) and remembering that A is positive $(A \geqslant \min_{\varphi_2} A > 0$, where $\min_{\varphi_2} A = J_* - 2\mu)$, we obtain the relation $(AJ_2 - B^2)\varphi_2^{\cdot 2} = 2eA - k^2$. Since $\min_{\varphi_2}(AJ_2 - B^2) = J_1 J_2 + m_2 l_1^2 J_2^{\circ} > 0$, it follows from the equation for $\phi_2^{\cdot 2}$ (by virtue of the reality of $\phi_2^{\cdot}$) that $2eA - k^2 \geq 0$. This inequality results in the following equivalent inequalities for all ϕ_2:

$$e \geqslant {}^1/_2 k^2 / A \geqslant {}^1/_2 k^2 / \min_{\varphi_2} A, \quad |k| \leqslant [2e(J_* - 2\mu)]^{1/2}$$

System (1.2) is solvable for the second derivatives $\phi_1^{\cdot\cdot}$, $\phi_2^{\cdot\cdot}$, since the quadratic form (1.1) is strictly positive-definite; the corresponding determinant Δ in (1.2) equals

$$\Delta = J_1 J_2 + m_2 l_1^2 J_2^{\circ} + \mu^2 \sin^2 \varphi_2 > 0 \quad (J_2 = J_2^{\circ} + m_2 l_2^2)$$

Using the first integrals of E (1.1) and K_1 (1.2), we obtain a relation between ϕ_2 and $\phi_2^{\cdot}$:

$$\frac{1}{2} \varphi_2^{\cdot 2} = \frac{2e}{I} \left[\frac{D - (1 - \cos \varphi_2)}{1 - \lambda^2 \cos^2 \varphi_2} \right]$$

$$I = J_2 (J_1 + m_2 l_1^2) \mu^{-1}, \quad D = 1 + {}^1/_2 (J_* - {}^1/_2 k^2 e^{-1}) \mu^{-1} \tag{1.3}$$

$$\lambda^2 = \mu I^{-1} = \mu^2 [J_2 (J_1 + m_2 l_1^2)]^{-1} \quad (0 \leqslant D < \infty, \ 0 < \lambda^2 < 1)$$

Passing to the limit as $\mu \to 0$, we find in accordance with the above that $\phi_2^{\cdot} = \text{const}$.

Equation (1.3) describes the motion of an equivalent system of the "pendulum type" with a variable (2π-periodic dependence) inertia characteristic $J(\phi_2)$. The Lagrange function has the form L = E - U, in which the kinetic energy $E = {}^1/_2 J(\varphi_2) \varphi_2^{\cdot 2}$, the potential energy $U = 2e(1 - \cos \phi_2)$, and $J(\varphi_2) = I(1 - \lambda^2 \cos^2 \varphi_2)$; for $\lambda^2 = 0$ we obtain the ordinary mathematical (or physical) pendulum. The corresponding equation of motion is obtained by differentiating expression (1.3) with respect to t. Relative oscillations in ϕ_2 take place in the system when $0 \leq D < 2$ and rotations when $D > 2$; the value $D = 2$ corresponds to motion along the separatrix.

To investigate relative motions, it is convenient to introduce the new dimensionless argument θ into (1.3):

$$\frac{1}{2} \varphi_2'^2 = \frac{D - (1 - \cos \varphi_2)}{1 - \lambda^2 \cos^2 \varphi_2}, \quad \varphi_2' = \frac{d\varphi_2}{d\theta}, \quad \theta = \left(\frac{2e}{I} \right)^{1/2} t \tag{1.4}$$

The phase paths that connect the variables ϕ_2 and ϕ_2' are analogous to the pendulum case. The problem of constructing periodic motions $\phi_2(\tau, D, \lambda^2)$ of the second link, i.e., solutions of Eq. (1.4), arises. Relation (1.4) describes the following motions:

symmetric oscillations [3], which occur when $0 \le D < 2$:

$$\theta-\theta_0=\int_{\varphi_2^0}^{\varphi_2}\frac{d\varphi}{\varphi_2'(\varphi, D, \lambda^2)}, \quad \varphi_2'=\pm\left[2\frac{D-(1-\cos\varphi_2)}{1-\lambda^2\cos^2\varphi_2}\right]^{1/2}$$

$$\varphi_2=\varphi_2(\psi, D, \lambda^2)\equiv\varphi_2(\psi+2\pi, D, \lambda^2), \quad |\varphi_2|\leqslant\varphi_2^*(D)$$

$$\varphi_2(\psi, D, \lambda^2)\equiv-\varphi_2(-\psi, D, \lambda^2), \quad \psi=\omega_2(D, \lambda^2)(\theta-\theta_0)+\psi^0 \tag{1.5}$$

$$\Theta_2(D, \lambda^2)=\oint\frac{d\varphi}{\varphi_2'}=2^{1/2}\int_{-\varphi_2^*}^{\varphi_2^*}\left[\frac{1-\lambda^2\cos^2\varphi}{D-(1-\cos\varphi)}\right]^{1/2}d\varphi$$

$$\omega_2=2\pi/\Theta_2, \quad \varphi_2^*(D)=\arccos(1-D)=2\arcsin(\tfrac{1}{2}D)^{1/2}$$

monotonic rotations [4,5] (Poincare-periodic solutions of the second form [4]) when $D > 2$ (for the sake of argument, in the positive direction):

$$\varphi_2=\varphi_2(\psi, D, \lambda^2)=\psi+\varphi_2^*(\psi, D, \lambda^2), \quad |\varphi_2^{*\prime}|<\omega_2$$

$$\varphi_2(\psi+2n\pi, D, \lambda^2)\equiv\varphi_2(\psi, D, \lambda^2)+2n\pi, \quad n=0, \pm1, \pm2, \ldots$$

$$\varphi_2^*(n\pi, D, \lambda^2)\equiv0, \quad \varphi_2(\psi, D, \lambda^2)\equiv-\varphi_2(-\psi, D, \lambda^2) \tag{1.6}$$

$$\Theta_2(D, \lambda^2)=\int_0^{2\pi}\frac{d\varphi}{\varphi_2'(\varphi, D, \lambda^2)}, \quad \varphi_2'=2^{1/2}\left[\frac{D-(1-\cos\varphi_2)}{1-\lambda^2\cos\varphi_2}\right]^{1/2}$$

$$\psi=\omega_2(\theta-\theta_0)+\psi^0; \quad \omega_2=2\pi/\Theta_2\to0, \ D\to2$$

Substitution of ω_2 $(\varphi_2'\to-\varphi_2')$ for ω_2 results in rotations in the negative direction $(\psi'<0, \varphi_2'<0)$; ω_2 and Θ_2 are known as the frequency and period of the rotations, respectively.

We note that the present system on integral manifold (1.2) pertains at small $D < 2$ to the case that generalizes the Lyapunov system [3] and when $D > 2$ to rotating systems, which were investigated by asymptotic methods with $\lambda^2 = 0$ in [5-8]. Rotational solutions were constructed in [9] with $D>\max_{\varphi_2}|U|, \lambda^2=0$ for an arbitrary periodic potential $U(\varphi_2)\equiv U(\varphi_2+2\pi)$.

The problem of analytic construction of oscillatory and rotational relative motions of the second link on the basis of expression (1.4) poses itself first. It is then necessary to use the integrals (1.2) to construct the motions of the second link and, finally, to find the motion of an arbitrary point of the system, for example, the end of the second link, on the plane O_1xy of the Cartesian variables.

2. CONSTRUCTION OF RELATIVE OSCILLATORY MOTIONS OF THE SECOND LINK

Let us give more detailed consideration to relations (1.5), which define closed phase paths on the (ϕ_2, ϕ_2') plane, vibrational motions $\varphi_2(\psi, D, \lambda^2)$, $\varphi_2'(\psi, D, \lambda^2)=\omega_2\partial\varphi_2/\partial\psi$, and their amplitudes $\phi_2^*(D)$, periods $\Theta_2(D, \lambda^2)$, and frequencies $\omega_2(D, \lambda^2)$. To express Θ_2, we obtain after the standard substitution $D=2\varkappa^2$, $\sin(\varphi_2/2)=\varkappa\sin\gamma$, $|\gamma|\leqslant\pi/2$, where $\kappa = \sin(\phi_2^*/2)$

$$\Theta_2=\Theta_2^*(\varphi_2^*, \lambda^2)=2\int_0^{\varphi_2^*}\left[\frac{1-\lambda^2\cos^2\varphi}{\varkappa^2-\sin^2(\varphi/2)}\right]^{1/2}d\varphi=$$

$$=4\int_0^{\pi/2}[1-\lambda^2(1-2\varkappa^2\sin^2\gamma)^2]^{1/2}(1-\varkappa^2\sin^2\gamma)^{-1/2}d\gamma=\sum_{n=0}^{\infty}\Theta_\varkappa^{(n)}(\lambda^2)\varkappa^{2n}= \tag{2.1}$$

$$= \sum_{m=0}^{\infty} \Theta_\lambda^{(m)}(\varkappa^2)\lambda^{2m} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \Theta_{\varkappa\lambda}^{(n,m)} \varkappa^{2n}\lambda^{2m} \tag{2.1}$$

For n = 0 we obtain an expression $\Theta_\varkappa^{(0)}(\lambda^2)$ for the period of small oscillations of the second link; with increasing λ^2, i.e., μ (see (1.3)), the period decreases; this also follows from (2.1). The subsequent coefficients $\Theta_\varkappa^{(n)}(\lambda^2)$, $n \geqslant 1$ are obtained in the form of elementary expressions by expanding the second representation of (2.1) for Θ_2 in powers of $(\kappa^2)^n$ and integration of $\sin^{2n}\gamma$ [10]; for example, we have for n = 0, 1, 2

$$\Theta_\varkappa^{(0)}(\lambda^2) = 2\pi(1-\lambda^2)^{1/2},\ \Theta_\varkappa^{(1)}(\lambda^2) = {}^1/_2\pi(1+3\lambda^2)(1-\lambda^2)^{-1/2}$$
$$\Theta_\varkappa^{(2)}(\lambda^2) = (3\pi/32)(3-14\lambda^2-5\lambda^4)(1-\lambda^2)^{-3/2}$$

We note that the known expression for the pendulum is obtained in the limit as $\lambda^2 \to 0$: $\Theta_2^*(\varphi_2^*, 0) = 4K(\varkappa)$, $0 \leqslant \varkappa < 1$, where K is a complete elliptic integral of the first kind, for which we have the expansion $\Theta_2^*(\varphi_2^*, 0) = 2\pi + {}^1/_2\pi\varkappa^2 + (9/32)\pi\varkappa^4 + \ldots$ (see [10,11]).

For $\lambda = 0$, $\kappa \sim 1$, the oscillatory motions of the second link are described by Jacobi elliptic functions [11,12]:

$$\varphi_2(\psi_0, \varkappa^2, 0) = 2\arcsin(\varkappa\,\mathrm{sn}(\psi_0, \varkappa)),\ 0 \leqslant \varkappa < 1$$
$$\psi_0 = \omega_{20}(\varkappa)(\vartheta - \vartheta_0) + \psi^0,\ \omega_{20}(\varkappa) = 2\pi/\Theta_2^*(\varphi^*, 0) = \pi/(2K(\varkappa))$$

On the basis of this generating solution we can construct the sought periodic solution $\phi_2(\psi, \kappa^2, \lambda^2)$ by perturbation methods (powers of the small parameter $\lambda^2 > 0$). With the substitutions indicated above, we obtain the relations

$$\gamma' = \frac{(1-\varkappa^2\sin^2\gamma)^{1/2}}{[1-\lambda^2(1-2\varkappa^2\sin^2\gamma)^2]^{1/2}} = \frac{(1-\varkappa^2\sin^2\gamma)^{1/2}}{1+\lambda^2\Gamma(\lambda^2, \varkappa^2\sin^2\gamma)}$$

$$\varphi_2 = 2\arcsin(\varkappa\sin\gamma) \quad (\gamma_0 = \mathrm{am}(\psi_0, \varkappa),\ \lambda^2 = 0)$$

We convert and integrate the resulting equation and reduce it to an implicit relation for determination of γ:

$$\gamma = \mathrm{am}(\Psi, \varkappa),\ \Psi = \psi - \lambda^2\Pi(\gamma, \varkappa^2, \lambda^2)$$
$$\lambda^2\Pi = \lambda^2\omega_2\Gamma + \omega_2 - \omega_{20},\ \omega_2 = 2\pi/\Theta_2^*(\varphi_2^*, \lambda^2)$$
$$\gamma_{(i+1)} = \mathrm{am}(\Psi_{(i-1)}, \varkappa),\ \Psi_{(i+1)} = \psi - \lambda^2\Pi(\gamma_{(i)}, \varkappa^2, \lambda^2) \tag{2.2}$$
$$\gamma_{(1)} = \mathrm{am}(\psi - \lambda^2\Pi(\gamma_{(0)}, \varkappa^2, 0), \varkappa),\ \gamma_{(0)} = \mathrm{am}(\psi, \varkappa),\ i = 1, 2, \ldots$$

Subsequent approximations (2.2) converge absolutely uniformly as $i \to \infty$ to the sought function $\gamma_*(\psi, \kappa^2, \lambda^2)$, and $\varphi_{2(i)} \to \varphi_{2*}(\psi, \varkappa^2, \lambda^2) = 2\arcsin(\varkappa\,\mathrm{sn}(\Psi_*, \varkappa))$, $\Psi_{(i)} \to \Psi_*$.

To construct the oscillations of the second link about the equilibrium position $\phi_2 = \phi_2' = 0$, we apply a procedure similar to the approach developed by A. M. Lyapunov to the systems that bear his name [3]. The corresponding equation and initial conditions have the form

$$(1-\lambda^2\cos^2\varphi_2)\varphi_2'' + {}^1/_2\lambda^2\varphi_2'^2\sin 2\varphi_2 + \sin\varphi_2 = 0$$
$$|\varphi_2| \leqslant \varphi_2^*(\delta) = 2\arcsin(\delta/2^{1/2}) \tag{2.3}$$
$$\varphi_2(0) = \varphi_2^*(\delta),\ \varphi_2'(0) = 0,\ \delta = D^{1/2} < 2^{1/2}$$

Putting $\phi_2 = \delta\xi$ in (2.3), where ξ is a new unknown variable, and introducing the perturbed time τ, we obtain the quasi-linear equation

$$\frac{d^2\xi}{d\tau^2}+h^2\xi=-\frac{\delta}{2}\lambda^2\left(\frac{d\xi}{d\tau}\right)^2\frac{\sin 2\delta\xi}{1-\lambda^2\cos^2\delta\xi}+\delta^2h^2F_\xi(\xi,\delta^2,\lambda^2)$$
$$\tau=\theta/(h(1-\lambda^2)^{1/2}),\ h=h(\delta^2,\lambda^2)=h_0+\delta^2h_1+\ldots+\delta^{2n}h_n+\ldots$$
$$h=\Theta_2/(2\pi(1-\lambda^2)^{1/2}),\ \xi=\xi(\tau,\delta^2,\lambda^2)\equiv\xi(\tau+2\pi,\delta^2,\lambda^2)$$
$$\xi(0,\delta^2,\lambda^2)=\varphi_2^*/\delta=(2/\delta)\arcsin(\delta/2^{1/2}),\ \xi'(0,\delta^2,\lambda^2)\equiv 0$$
$$\delta^2F_\xi(\xi,\delta^2,\lambda^2)\equiv\delta^{-1}[\delta\xi(1-\lambda^2)^{-1}-\sin\delta\xi(1-\lambda^2\cos^2\delta\xi)^{-1}]$$
$$\xi(\tau,\delta^2,\lambda^2)=\xi_0(\tau,\lambda^2)+\delta^2\xi_1(\tau,\lambda^2)+\ldots+\delta^{2n}\xi_n(\tau,\lambda^2)+\ldots$$

Here the parameter h is so chosen that the variables ξ, ξ' will be 2π-periodic with respect to τ. The coefficients h_i, i = 1, 2, ..., and h can be determined either during construction of the solution or according to (2.4), where Θ_2 is determined by relation (2.1), in which $\kappa^2 = \delta^2/2$. Substitution of the expansions for the unknown h, ξ in powers of $(\delta^2)^n$ into (2.4) and equating the coefficients with consideration of the 2π-periodicity conditions, we obtain the expressions sought for $h_n(\lambda^2)$, $\xi_n(\tau,\lambda^2)$:

$$h_0=1,\ h_1=(1/8)(1+3\lambda^2)/(1-\lambda^2),\ h_2=(3/256)(3-14\lambda^2-5\lambda^4)/(1-\lambda^2)^2,\ \ldots$$
$$\xi_0=2^{1/2}\cos\tau,\ \xi_1=(2^{1/2}/96)[3(3+\lambda^2)\cos\tau-(1+11\lambda^2)\cos 3\tau]/(1-\lambda^2)$$
$$\xi_2=(2^{1/2}/46080)[(1875-3390\lambda^2+85275\lambda^4)\cos\tau+ \tag{2.5}$$
$$+(-180+2160\lambda^2+2340\lambda^4)\cos 3\tau+(9+702\lambda^2+$$
$$+1449\lambda^4)\cos 5\tau]/(1-\lambda^2)^2,\ \ldots$$
$$\tau=(2\pi/\Theta_2)(\theta-\theta_0)=(\theta-\theta_0)/(h(1-\lambda^2)^{1/2})$$

It follows from (2.4) and (2.5) that $\xi_0(0,\lambda^2)+\delta^2\xi_1(0,\lambda^2)+\ldots\equiv\varphi_2^*(\delta)\delta^{-1}$. We note that the series (2.4) and (2.5) will converge uniformly if $D = \delta^2 \le c < 2$. Thus, a periodic solution that depends on the three arbitrary parameters e, k, θ_0 can be constructed with any predetermined degree of accuracy in the parameter D.

3. CONSTRUCTION OF ROTATIONAL MOTIONS OF THE SECOND LINK

As in Sec. 2, we devote a more detailed analysis to expression (1.6), which defines 2π-periodic phase paths $\phi_2'(\phi_2)$ on the phase plane (ϕ_2, ϕ_2'), monotonic rotational motions $\phi_2(\psi, D, \lambda^2)$ for which the velocity $\varphi_2'(\psi, D, \lambda^2)=\omega_2\partial\varphi_2/\partial\psi\geqslant\nu>0$ is strictly positive, the velocity extremes, the period $\Theta_2(D, \lambda^2)$, and the frequency $\omega_2 = 2\pi/\Theta_2$. To express Θ_2 we have

$$\Theta_2=\Theta_2(D,\lambda^2)=(2D)^{1/2}\int_0^\pi\left[\frac{1-\lambda^2\cos^2\varphi}{1-D^{-1}(1-\cos\varphi)}\right]^{1/2}d\varphi=$$
$$=\frac{4}{\varkappa}\int_0^{\pi/2}\left(\frac{1-\lambda^2\cos^2 2\gamma}{1-\varkappa^2\sin^2\gamma}\right)^{1/2}d\gamma=\frac{4}{\varkappa}\sum_{n=0}^{\infty}\Theta_\varkappa^{(n)}(\lambda^2)\varkappa^{2n}=\frac{4}{\varkappa}\sum_{m=0}^{\infty}\Theta_\lambda^{(m)}(\varkappa^2)\lambda^{2m}= \tag{3.1}$$
$$=\frac{4}{\varkappa}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\Theta_{\varkappa\lambda}^{(n,m)}\varkappa^{2n}\lambda^{2m},\quad \varkappa^2=\frac{2}{D}<1$$

In the limit as $\lambda \to 0$ we obtain for Θ_2 the expression $\Theta_2(D, 0) = (4/\varkappa)K(\varkappa)$ (see [12]), which corresponds to the rotations of the second link on uniform rotation of the first body (see the remarks following Eqs. (1.2)). We note that expansion in κ^2 or λ^2 result in complete elliptic integrals of the first and second kinds, while the double series (3.1) in κ^2 and λ^2 results in elementary integrals of $\sin^{2n}\gamma\cos^{2m}2\gamma$ [10]. As a result we obtain the following approximate expression for the period of "fast rotations" (at small D^{-1}, λ^2, $\lambda^2 \sim D^{-1}$):

$$\Theta_2(D,\lambda^2)=\pi(2D)^{1/2}h(D^{-1},\lambda^2),\ D=2\varkappa^{-2}$$

$$h=1+{}^1/_2(D^{-1}-{}^1/_2\lambda^2)+(1/16)(9D^{-2}-2D^{-1}\lambda^2-(3/4)\lambda^4)+ \tag{3.2}$$
$$+(1/64)(60D^{-3}-77D^{-2}\lambda^2+39D^{-1}\lambda^4-(25/2)\lambda^6)+O(D^{-4}+\lambda^8)$$

Let us consider positive-direction rotational motions that correspond to this case. We introduce the "fast" dimensionless time ϑ and modify expressions (1.4), (1.6):

$$d\vartheta/d\varphi_2=h(D^{-1},\lambda^2)+g(\varphi_2,D^{-1},\lambda^2),\ \vartheta=(2D)^{1/2}\theta$$

$$\mathrm{T}(\varphi_2,D^{-1},\lambda^2)=h+g\equiv[(1-\lambda^2\cos^2\varphi_2)/(1-D^{-1}(1-\cos\varphi_2))]^{1/2}$$

$$h=\langle\mathrm{T}\rangle_{\varphi_2}\equiv\frac{1}{2\pi}\int_0^{2\pi}\mathrm{T}(\varphi,D^{-1},\lambda^2)\,d\varphi\qquad(h(0,0)=1) \tag{3.3}$$

$$g=\mathrm{T}-\langle\mathrm{T}\rangle_{\varphi_2}=\mathrm{T}-h,\ \langle g\rangle_{\varphi_2}\equiv0,\ g=O(D^{-1}+\lambda^2)$$

Equation (3.3) is integrated by quadrature. Using the properties of the function g (smallness with respect to D^{-1}, λ^2 for all ϕ_2, 2π-periodicity and a zero mean value with respect to ϕ_2), we run a procedure of the method of perturbations. We shall seek a rotational solution $\phi_2(\psi, D^{-1}, \lambda^2)$ of the form (1.4) by expansions (or successive approximations) in powers of the small parameters D^{-1}; λ^2 ($\phi_2{}^\circ = 0$):

$$\varphi_2=\psi+G(\varphi_2,D^{-1},\lambda^2),\ \psi=\vartheta/h=\omega_2(\theta-\theta_0)+\psi^\circ$$

$$G(\varphi_2,D^{-1},\lambda^2)=\frac{1}{h}\int_0^{\varphi_2}g(\varphi,D^{-1},\lambda^2)\,d\varphi,\qquad G=O(D^{-1}+\lambda^2)$$

$$\varphi_2=\psi+G(\psi,D^{-1},\lambda^2)[1+g(\psi,D^{-1},\lambda^2)]+O(D^{-3}+\lambda^6)$$
$$g(\psi,D^{-1},\lambda^2)=-{}^1/_2D^{-1}\cos\psi-(1/4)\lambda^2\cos2\psi+$$
$$+(3/4)D^{-1}(\lambda^2/4-D^{-1})\cos\psi+(1/16)(D^{-1}+\lambda^2)(3D^{-1}+\lambda^2)\cos2\psi+ \tag{3.4}$$
$$+(1/16)D^{-1}\lambda^2\cos3\psi-(\lambda^4/64)\cos4\psi+O(D^{-3}+\lambda^6)$$
$$h(D^{-1},\lambda^2)G(\psi,D^{-1},\lambda^2)=-{}^1/_2D^{-1}\sin\psi-(\lambda^2/8)\sin2\psi+$$
$$+(3/4)D^{-1}(\lambda^2/4-D^{-1})\sin\psi+(1/32)(D^{-1}+\lambda^2)(3D^{-1}+\lambda^2)\sin2\psi+$$
$$+(1/48)D^{-1}\lambda^2\sin3\psi+(\lambda^2/256)\sin4\psi+O(D^{-3}+\lambda^6)$$

Thus, formulas (3.1)-(3.4) give us an approximate solution of the problem of "fast rotations" of the second link when the motion of the first (carrier) body is nearly uniform rotation (large moment of inertia J_1).

In the limit as $\lambda^2 \to 0$, the case $\lambda^2 \ll 1$, $D = 2\varkappa^{-2} > 2$ ($D \sim 2$) results in rotations of the pendulum, the motions of which is described by elliptic functions [11,12]:

$$\varphi_2(\psi_0,D^{-1},0)=2\,\mathrm{am}(\psi_0,\varkappa)=\psi_0+4\sum_{j=1}^{\infty}\frac{1}{j}\frac{q^j}{1+q^{2j}}\sin j\psi_0$$
$$\psi_0=\omega_2(D^{-1},0)(\theta-\theta_0)+\psi^\circ,\ \Theta_2(D^{-1},0)=4\varkappa^{-1}\mathrm{K}(\varkappa) \tag{3.5}$$
$$q=q(\varkappa)\equiv\exp[-\pi\mathrm{K}'(\varkappa)/\mathrm{K}(\varkappa)]<1,\ \mathrm{K}'(\varkappa)=\mathrm{K}(\varkappa'),\ \varkappa'=(1-\varkappa^2)^{1/2}<1$$

where am is the elliptic amplitude, K is a complete elliptic integral of the first kind, and κ is the modulus. At small λ^2, the autorotational motion (1.6) described by Eq. (1.3) is obtained with the aid of the Lyapunov-Poincare methods [3] or by construction following the procedure of [12]. We have

$$d\psi/d\varphi_2=1+g_\lambda(\varphi_2, D^{-1})+\lambda^2\delta_\lambda(\varphi_2, D^{-1}, \lambda^2)$$
$$\psi=\vartheta/h,\ g_\lambda(\varphi_2, D^{-1})=g(\varphi_2, D^{-1}, 0)/h(D^{-1}, 0) \tag{3.6}$$
$$\lambda^2\delta_\lambda(\varphi_2, D^{-1}, \lambda^2)\equiv g(\varphi_2, D^{-1}, \lambda^2)/h(D^{-1}, \lambda^2)-g_\lambda(\varphi_2, D^{-1})$$

The functions g, g_λ, δ_λ are 2π-periodic and have zero mean values with respect to ϕ_2. Integrating (3.6), we obtain a finite relation that determines ϕ_2 implicitly (see (3.5)):

$$\psi=\varphi_2+G_\lambda(\varphi_2, D^{-1})+\lambda^2\Delta_\lambda(\varphi_2, D^{-1}, \lambda^2)$$
$$\varphi_2^{(0)}(\psi, D^{-1})=2\,\mathrm{am}(\psi, \varkappa),\ \ \varphi_2^{(1)}(\psi, D^{-1}, \lambda^2)= \tag{3.7}$$
$$=2\,\mathrm{am}(\Psi^{(1)}, \varkappa),\ \ \Psi^{(1)}=\psi-\lambda^2\Delta_\lambda(\varphi_2^{(0)}, D^{-1}, 0)$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\varphi_2^{(i+1)}(\psi, D^{-1}, \lambda^2)=2\,\mathrm{am}(\Psi^{(i+1)}, \varkappa),\quad \Psi^{(i+1)}=\psi-\lambda^2\Delta_\lambda(\varphi_2^{(i)}, D^{-1}, \lambda^2)$$

Here am is the elliptic amplitude corresponding to the integral of the first kind $F(\kappa, \phi_2)$ [11]. Successive approximations $\phi_2^{(i)}$ of (3.7) converge uniformly as $i \to \infty$ to the sought rotational solution of Eq. (3.6) $\phi_2^{(*)}(\psi, D^{-1}, \lambda^2)$ in the form (1.6) if λ^2 is small ($\lambda^2 \leq c < 1$). The "rapid rotation" case is handled similarly, i.e., the expansion is constructed in powers of the parameter D^{-1} (or κ^2) for $\lambda^2 \sim 1$ ($\lambda^2 < 1$). The corresponding generating equation is determined similarly from (3.6) and (3.3):

$$d\psi/d\varphi_2=1+g_D(\varphi_2, \lambda^2)+D^{-1}\delta_D(\varphi_2, D^{-1}, \lambda^2)$$
$$\psi=\vartheta/h,\ g_D(\varphi_2, \lambda^2)=g(\varphi_2, 0, \lambda^2)/h(0, \lambda^2)=$$
$$=(1-\lambda^2\cos^2\varphi_2)^{1/2}/h(0, \lambda^2)-1,\quad h(0, \lambda^2)=(2/\pi)\,E(\lambda) \tag{3.8}$$
$$D^{-1}\delta_D(\varphi_2, D^{-1}, \lambda^2)\equiv g(\varphi_2, D^{-1}, \lambda^2)/h(D^{-1}, \lambda^2)-g_D(\varphi_2, \lambda^2)$$

It follows from (3.8) that the relation between the phase ψ and the variable $\phi_2^{\pm} = \phi_2 \pm \pi/2$ is given by an elliptic integral of the second kind: $\psi = E(\lambda, \phi_2^{\pm})$, which we invert to write $\varphi_2=\mp\pi/2+\mathrm{am}_2(\psi, \lambda)$. Integrating (3.8), we obtain an implicit equation for ϕ_2 and its solution by successive approximations according to (3.8):

$$\psi=\varphi_2+G_D(\varphi_2, \lambda^2)+D^{-1}\Delta_D(\varphi_2, D^{-1}, \lambda^2)$$
$$\varphi_2^{(0)}(\psi, \lambda^2)=\mp\pi/2+\mathrm{am}_2(\psi, \lambda)$$
$$\varphi_2^{(1)}(\psi, D^{-1}, \lambda^2)=\mp\pi/2+\mathrm{am}_2(\Psi^{(1)}, \lambda),\ \ \Psi^{(1)}=\psi+D^{-1}\Delta_D(\varphi_2^{(0)}, 0, \lambda^2) \tag{3.9}$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\varphi_2^{(i+1)}(\psi, D^{-1}, \lambda^2)=\mp\pi/2+\mathrm{am}_2(\Psi^{(i+1)}, \lambda),\quad \Psi^{(i+1)}=\psi-D^{-1}\Delta_D(\varphi_2^{(i)}, D^{-1}, \lambda^2),$$
$$i=1, 2, \ldots$$

The successive approximations $\phi_2^{(i)}$ (3.9) converge uniformly as $i \to \infty$ to the sought rotational solution of Eq. (3.8) ($\phi_2^{(*)}(\psi, D^{-1}, \lambda^2)$ in the form (1.6), if D^{-1} is small ($D^{-1} \leq c < 2$). The case of small $D^{-1} \ll 1$ ("fast rotations" of the second link) but large $\lambda^2 \sim 1$ ($\lambda^2 < 1$) is not mechanically instructive. We note further that extremes of the angular velocity $\phi_2^{\cdot}(\phi_2)$ occur at $\phi_2 = 0, \pi$ (maximum and minimum, respectively, for positive rotation), as can be seen on direct differentiation of expression (1.3). Thus, a general approximate analytic solution $\phi_2(\psi, D^{-1}, \lambda^2)$ of (1.3) (or (1.4)) in the form of oscillations and rotations of the second link has been constructed (see (3.4), (3.7), (3.9)). This solution depends on the two motion integrals e and k and on the phase constant ($-\omega_2\theta^{\circ} + \psi^{\circ}$) as well as on the system parameters (see Sec. 1). It can be used as a basis for construction of a general solution for the variable ϕ_1, which determines the motion of the first, carrier, body according to (1.2).

4. DETERMINING THE MOTION OF THE FIRST (CARRIER) BODY

The motion of the first link with respect to the nonrotating O_1xy system is given by the angular variable ϕ_1. We use the motion integrals (1.1) and (1.2) to determine it. Let us first consider the general case $\mu \sim J_2, J_*$ $(0 < \mu, J_2 < J_*)$; we have the expression

$$\varphi_1=\varphi_1(\varphi_2)=\varphi_1^\circ+\int_{\varphi_2^\circ}^{\varphi_2}\frac{k-(J_2+\mu\cos\varphi)\varphi_2^{\cdot}(\varphi)}{(J_*+2\mu\cos\varphi)\varphi_2^{\cdot}(\varphi)}d\varphi \tag{4.1}$$

in which $\phi_2^{\cdot}$ is determined according to (1.3) and the $\phi_{1,2}^\circ$ are certain fixed values of the variables $\phi_{1,2}$ that correspond to one another (for example, the initial values). Relation (4.1) states the relation between these variables, but it is inconvenient for analysis because the integral contains an essential singularity: $\phi_2^{\cdot}(\pm\phi_2^*) = 0$, and the variable ϕ_2 is nonmonotonic for oscillatory motions, see Sec. 2. It is more convenient to convert to the time variable t or to the phase ψ_2 of the second link:

$$\varphi_1^{\cdot}(\psi_2)=\frac{k-[J_2+\mu\cos\varphi_2(\psi_2)]\varphi_2^\circ(\psi_2)}{J_*+2\mu\cos\varphi_2(\psi_2)}\qquad(\psi_2\equiv\psi)$$

$$\psi_2=\omega_2(\theta-\theta_0)+\psi_2^\circ=\Omega_2(t-t_0)+\psi_2^\circ,\ \Omega_2=\omega_2(D,\lambda^2)(2e/I)^{1/2}$$

Separating the mean part and the variable part with zero mean value, we bring the angular velocity $\phi_1^{\cdot}(\psi_2)$ to a form convenient for integration:

$$\varphi_1^{\cdot}=\Omega_1(e,k,P)+\Delta_1(\psi_2,e,k,P),\ \langle\Delta_1\rangle_{\psi_2}\equiv0$$

$$\varphi_1=\varphi_1(\psi_1,\psi_2)=\varphi_1^\circ+\psi_1+\Phi_1(\psi_2,e,k,P)$$

$$\Omega_1=\frac{1}{2\pi}\int_0^{2\pi}\varphi_1^{\cdot}(\psi,e,k,P)d\psi,\qquad\psi_1=\Omega_1(t-t_0)$$

$$\Phi_1=\int_{t_0}^{t}\Delta_1(\psi_2',e,k,P)dt',\qquad\psi_2=\Omega_2(t-t_0)+\psi_2^\circ$$

Here P is the set of values of the system parameters and Δ_1, Φ_1 are 2π-periodic functions of the phase ψ_2.

Thus, the motion is one-frequency if $\Omega_1 = 0$: $\phi_{1,2} = \phi_{1,2}(\psi_2)$, with the variable ϕ_1 oscillating, while ϕ_2 may be either oscillating or rotating. In this case, the paths $(\phi_1(\psi_2), \phi_2(\psi_2))$ are closed. In the inertial O_1xy space, points of the first carrier body, for example those lying on the O_1O_2 axis, describe oscillatory motions along an arc of radius L_1 whose paths are closed for $\Phi_{1\,max}-\Phi_{1\,min}\geqslant2\pi$ $(x=L_1\cos\varphi_1,\ y=L_1\sin\varphi_1)$. Points of the carried second link described relative oscillations or rotations with the same period, and this results in closed paths. For points on the O_2C axis, for example, we obtain $x=L_2\cos(\varphi_1+\varphi_2)+l_1\cos\varphi_1, y=L_2\sin(\varphi_1+\varphi_2)+l_1\sin\varphi_1$.

Further, if $\Omega_1 \neq 0$, the paths are generally nonperiodic (nonclosed), and the motion is two-frequency. Quasi-periodic motions occur for Ω_1/Ω_2 irrational. The closed-path condition and the expression for the corresponding period T_Σ have the form

$$\Omega_1/\Omega_2=p/q,\ T_\Sigma=pT_1=qT_2,\ T_{1,2}=2\pi/\Omega_{1,2}$$
$$q\Omega_1(e,k,P)=p\Omega_2(e,k,P) \tag{4.2}$$

Here p, q = ±1, ±2, ... are mutually prime integers. Here it is necessary to consider the limits on the set of values of e, k, and P, for example $e\geqslant{}^1/_2k^2(J_*-2\mu)^{-1}$, see Sec. 1.

Let us now consider the case of small values of the parameter μ $(\mu \ll J_2)$, which characterizes the interaction of the links. In the limit at $\mu = 0$ we have the expressions

$$\varphi_2^{\cdot}=\Omega_2^{(0)}(e,k,P)=\pm f(e,k,P)=\varphi_2^{\cdot\circ}=\mathrm{const}$$
$$\varphi_2^{\cdot}=\Omega_1^{(0)}(e,k,P)=(k\mp fJ_2)J_*^{-1}=\varphi_1^{\cdot\circ}=\mathrm{const}$$
$$f=f(e,k,P)=(2eJ_*-k^2)^{1/2}[J_2(J_*-J_2)]^{-1/2}$$

Hence follows that each of the variables ϕ_1, ϕ_2 may be either constant (if $\varphi_1^{\cdot\circ}=0, \varphi_2^{\cdot\circ}=0$), or rotating (if $\varphi_1^{\cdot\circ}\neq 0,\ \varphi_2^{\cdot\circ}\neq 0$). The condition (4.2) for periodicity (closing) of the paths reduces to relationships in terms of the integrals e, k and the angular velocities $\phi_{1,2}^{\cdot\circ}$, respectively:

$$k=\pm(J_2+J_*p/q)f(e,k,P),\ k^2\leqslant 2eJ_*,\ \varphi_1^{\cdot\circ}/\varphi_2^{\cdot\circ}=p/q$$
$$k^2/(2eJ_*)=[1+\rho(1-\rho)(\rho+p/q)^{-2}]^{-1},\ 0<\rho=J_2/J_*<1$$

In the first approximation in μ for $\Omega_2^{(0)}\neq 0$, we obtain expressions for $\phi_1^{\cdot}$, $\phi_2^{\cdot}$ and the closed-path condition:

$$\varphi_1^{\cdot}=\Omega_1^{(0)}+\mu(N^2/\Delta J)(\cos\varphi_2^{\circ}-\cos\varphi_2^{(0)})/\Omega_2^{(0)}$$
$$\varphi_2^{\cdot}=\Omega_2^{(0)}-\mu(N^2/\Delta J+\Omega_1^{(0)2}/J_2)(\cos\varphi_2^{\circ}-\cos\varphi_2^{(0)})/\Omega_2^{(0)}$$
$$q\left(\Omega_1^{(0)}+\frac{\mu}{\Delta J}\frac{N^2}{\Omega_2^{(0)}}\cos\varphi_2^{\circ}\right)=p\left[\Omega_2^{(0)}-\mu\left(\frac{N^2}{\Delta J\Omega_2^{(0)}}+\frac{\Omega_1^{(0)2}}{J_2\Omega_2^{(0)}}\right)\cos\varphi_2^{\circ}\right]$$
$$N^2=2\Omega_2^{(0)2}+\Omega_2^{(0)}\Omega_1^{(0)}+\Omega_1^{(0)2},\quad \Delta J=J_*-J_2>0$$
$$J_*=J_1+J_2+m_2l_1^2,\ J_2=J_2^{\circ}+m_2l_2^2,\ \mu=m_2l_1l_2$$

Thus, general periodic and quasi-periodic oscillatory and rotational motions of the unperturbed conservative system described in Sec. 1 were constructed in Secs. 2-4. They are of applied interest for investigation of the motions of a plane two-link under the actin of perturbing and control inputs (for example, in the hinges O_1, O_2 and from the external environment). These motions are Lyapunov unstable in either direction, i.e., at $t\gtrless 0$. The above analyses can be extended directly to a free system of coupled rigid bodies [1].

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