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EVOLUTION OF ROTATION OF A NEARLY DYNAMICALLY SPHERICAL TRIAXIAL SATELLITE UNDER THE ACTION OF GRAVITATIONAL AND LIGHT TORQUES

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We study the evolution of rotation of a rigid body (a Sun satellite moving in an elliptic orbit with arbitrary eccentricity) subjected to moments of gravitational forces and light pressure. The body is assumed to be nearly dynamically spherical, and its surface is assumed to be a surface of revolution, which allows one to approximate the light pressure torque coefficient by a finite trigonometric polynomial. In the first approximation of the averaging method, we obtain new qualitative effects of satellite rotation relative to its center of mass.

1. PRIMARY ASSUMPTIONS AND STATEMENT OF THE PROBLEM

We consider the motion of a satellite or a spacecraft relative to the center of mass under the joint action of the moments of light pressure forces and gravitational attraction forces. The rotational motions are studied in the framework of models describing the dynamics of rigid bodies whose centers of mass move in elliptic orbits around the Sun. The dynamic problems are generalized and complicated by taking into account various perturbing factors and remain topical nowadays. Rotational motions of bodies relative to the center of mass under the action of perturbing torques of various nature (gravitational, light pressure, etc.) were studied in a close manner in numerous papers (see [1–10] and the references therein).

We introduce three right Cartesian frames with origin at the center of inertia of the satellite [1, 2]. The frame $OXYZ$ moves progressively in the Sun orbit together with the satellite; the Y-axis is parallel to the normal to the orbit plane, the Z-axis is parallel to the direction of the position vector of the orbit at the perihelion, and the X-axis is the direction of the velocity vector of the center of mass at the perihelion.

The position of the angular momentum vector $\mathbf{L}$ in the frame $OXYZ$ is determined by the angles ρ and σ, as was shown in [1–3]. (Here $\mathbf{L}$ is the angular momentum of the body relative to its center of mass.) To construct the frame OL_1L_2L fixed to the vector $\mathbf{L}$, in the OYL-plane we draw the L_1-axis perpendicularly to the vector $\mathbf{L}$ and forming an obtuse angle with the Y-axis. The L_2-axis complements the L_1- and L-axes to a right frame. We let the axes of the body-fixed frame $Oxyz$ coincide with the central principal axes of inertia. The mutual position of the central principal axes of inertia and the L-, L_1-, L_2-axes is determined by the Euler angles [1–3]. The direction cosines (α_{ij}) of the axes Ox, Oy, Oz with respect to the frame OL_1L_2L are expressed via the Euler angles φ, ψ, θ by well-known formulas [1].

We neglect the moments of all forces except for the gravitational forces and the light pressure forces. In [1], a comparative estimate of gravitational and light pressure torques is given for a Sun satellite. In general situation, it was shown that the light pressure torque is several orders of magnitude greater than the gravitational torque. In the problem considered below, the light pressure torque is assumed to be of the same order of magnitude ε as the gravitational torque. For example, this can be achieved by an appropriate mass distribution and an appropriate shape of the body.

The gravitational torque acting on the satellite from the Sun side has the form [1, 3]

$$\mathbf{M}_g = \frac{3\mu}{R^3}\left((C-B)\gamma'\gamma'', (A-C)\gamma\gamma'', (B-A)\gamma'\gamma\right), \quad R = |\mathbf{R}|. \tag{1.1}$$

Here μ is the gravitational parameter of the Sun; γ, γ', and γ'' are the cosines of the angles between the position vector **R** and the x-, y-, and z-axes.

We assume that the surface of the spacecraft is a surface of revolution and the unit vector **k** of the symmetry axis is directed along the axis Oz. As was shown in [1, 4, 5], the light pressure torque $\mathbf{M}_c$ acting on the satellite in this case is given by the formula

$$\mathbf{M}_c = a_c(\varepsilon_s)\frac{R_0^2}{R^2}\mathbf{e_r} \times \mathbf{k},$$
$$a_c(\varepsilon_s)\frac{R_0^2}{R^2} = p_c S(\varepsilon_s) Z_0'(\varepsilon_s), \quad p_c = \frac{E_0}{c}\left(\frac{R_0}{R}\right)^2. \tag{1.2}$$

Here $\mathbf{e_r}$ is the unit vector directed along the position vector of the orbit; ε_s is the angle between the directions $\mathbf{e}_r$ and **k**, so that $|\mathbf{e_r} \times \mathbf{k}| = \sin\varepsilon_s$; R is the current distance from the Sun center to the satellite center of mass; R_0 is a fixed value of R, say, at the initial time; $a_c(\varepsilon_s)$ is the light pressure torque coefficient determined by the properties of the surface; S is the area of the "shadow" on the plane normal to the flow; Z_0' is the distance from the center of mass to the center of pressure; p_c is the value of light pressure at the distance R from the Sun center; c is the light velocity; and E_0 is the value of the light pressure energy flux at the distance R_0 from the Sun center. If R_0 is the radius of Earth orbit, then $p_{c0} = 4.64 \cdot 10^{-6}$ Pa.

In what follows, we assume [1] that the function a_c has the form $a_c = a_c(\cos\varepsilon_s)$ and approximate this function by polynomials in powers of $\cos\varepsilon_s$. The light pressure torque has a force function depending only on the position of the symmetry axis in space [1]. We represent the function $a_c(\cos\varepsilon_s)$ as [9]

$$a_c = a_{0c} + a_{1c}\cos\varepsilon_s + \ldots + a_N\cos^N\varepsilon_s. \tag{1.3}$$

If the force function exists, then the equations of perturbed motion of the satellite in the variables L, ρ, σ, φ, ψ, θ have the form [3]

$$\dot\sigma = (L\sin\rho)^{-1}\frac{\partial U}{\partial \rho}, \quad \dot\rho = -(L\sin\rho)^{-1}\frac{\partial U}{\partial\sigma} + \cot\rho L^{-1}\frac{\partial U}{\partial\psi}, \quad \dot L = \frac{\partial U}{\partial\psi},$$
$$\dot\theta = L\sin\theta\sin\varphi\cos\varphi(A^{-1} - B^{-1}) - (L\sin\theta)^{-1}\frac{\partial U}{\partial\varphi} + \cot\theta L^{-1}\frac{\partial U}{\partial\psi},$$
$$\dot\varphi = L\cos\theta(C^{-1} - A^{-1}\sin^2\varphi - B^{-1}\cos^2\varphi) + (L\sin\theta)^{-1}\frac{\partial U}{\partial\theta}, \tag{1.4}$$
$$\dot\psi = L(A^{-1}\sin^2\varphi + B^{-1}\cos^2\varphi) - L^{-1}\left(\frac{\partial U}{\partial\rho}\cot\rho + \frac{\partial U}{\partial\theta}\cot\theta\right).$$

The force function U depends on time t via the true anomaly $\nu(t)$ and on the direction cosines of the axis Oz in the frame $OXYZ$; it has the form $U = U(\nu(t), \alpha_3, \beta_3, \gamma_3)$.

One should supplement Eqs. (1.4) with the equation describing the variation of the true anomaly in time,

$$\frac{d\nu}{dt} = \omega_0(1-e^2)^{-3/2}(1+e\cos\nu)^2,$$
$$\omega_0 = \frac{2\pi}{T_0} = \left[\frac{\kappa(1-e^2)^3}{P^3}\right]^{1/2}. \tag{1.5}$$

Here ω_0 is the average angular velocity of motion of the center of mass in an elliptic orbit; T_0 is the orbital period of the satellite; e and P are, respectively, the eccentricity and the focal parameter of the orbit; and κ is the product of the universal gravitational constant by the Sun mass.

We assume that the force function consists of two terms due to the gravitational torque and the light pressure torque, $U = U_g + U_c$. The force function due to the gravitational moment can be written as

$$U_g = \frac{3\mu}{2R^3}[(A-B)\gamma'^2 + (A-C)\gamma''^2],$$
$$\gamma' = \alpha_2\sin\nu + \gamma_2\cos\nu, \quad \gamma'' = \alpha_3\sin\nu + \gamma_3\cos\nu. \tag{1.6}$$

The light pressure torque (1.2) corresponds to the force function

$$U_c(\cos\varepsilon_s) = -\frac{R_0^2}{R^2}\int a_c(\cos\varepsilon_s)\, d(\cos\varepsilon_s).$$

We first consider the case of the "trigonometric monomial"

$$a_c(\cos\varepsilon_s) = a_n \cos^n \varepsilon_s. \tag{1.7}$$

The force function in this case has the form

$$U_c = -\frac{a_n R_0^2}{(n+1)R^2}\cos^{n+1}\varepsilon_s, \qquad \cos\varepsilon_s = \gamma_3\cos\nu + \alpha_3\sin\nu. \tag{1.8}$$

The direction cosines α_3 and γ_3 are expressed via ρ, σ, θ and ψ by well-known formulas [1]. We assume that the angular velocity of the satellite motion relative to the center of mass is significantly larger than the angular velocity ω_0 of orbital motion; i.e., we assume that $\varepsilon = \omega_0/\omega \sim A\omega_0/G \ll 1$.

We assume that the central principal moments of inertia of the satellite are close to one another and can be represented as

$$A = J_0 + \varepsilon A_1, \quad B = J_0 + \varepsilon B_1, \quad C = J_0 + \varepsilon C_1, \tag{1.9}$$

where $0 < \varepsilon \ll 1$ is a small parameter. We also assume that $a_0 \sim \varepsilon$, $a_1 \sim \varepsilon$, $\ldots$, $a_N \sim \varepsilon$; i.e., the light pressure torques have the same order of magnitude ε as the gravitational as well as gyroscopic torques. It follows from (1.7) that $U_c \sim \varepsilon$. Moreover, for the force function of the gyroscopic torque one has $U_g \sim \varepsilon$; i.e. $A - C = (A_1 - C_1)\varepsilon$ and $A - B = (A_1 - B_1)\varepsilon$.

We study the solution of system (1.4), (1.5) for small ε on a large time interval $t \sim \varepsilon^{-1}$. The error in the averaged solution for slow variables is $O(\varepsilon)$ on the time interval in which the body makes ε^{-1} rotations. The independent averaging over ψ and ν is performed just as in the nonresonance cases [2].

2. TRANSFORMATION OF THE EXPRESSION FOR THE FORCE FUNCTION, THE AVERAGING PROCEDURE, AND THE CONSTRUCTION OF THE FIRST APPROXIMATION SYSTEM

Consider the unperturbed motion ($\varepsilon = 0$) for the case in which Eqs. (1.4), (1.5) describe the motion of a spherically symmetric body and the gravitational torques (1.1), as well as the light and gyroscopic torques (1.2), are zero. It follows from system (1.4) that ρ, σ, L, θ, and φ are constant and

$$\psi = \frac{L}{J_0}t + \psi_0, \quad \psi_0 = \text{const}, \tag{2.1}$$

which corresponds to the uniform rotation of the satellite about the vector of the angular momentum **L** which moves progressively. For small $\varepsilon \neq 0$, system (1.4), (1.5) of seven equations with (1.9) taken into account contains the slow variables ρ, σ, L, θ, φ and the fast variables ψ and ν. To obtain the solution in the first approximation, it suffices to average the right-hand sides of Eqs. (1.4) by substituting ν from the solution of (1.5) and ψ from the solution of (2.1) into these equations. We assume that no resonance relations of the form $m_1\omega_0 + n_1 L J_0^{-1} \neq 0$, where m_1 and n_1 are arbitrary integers, hold for the frequencies ω_0 and LJ_0^{-1}.

Then the time averaging of the force functions can be replaced by the following independent averaging over the variables ψ and $\nu(t)$:

$$\bar{U} = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} U\,d\psi\,dt = \frac{1}{(2\pi)^2}\int_0^{2\pi}\left(\int_0^{2\pi} U\,d\psi\right)\frac{dt}{d\nu}\,d\nu = \frac{1}{(2\pi)^2}\int_0^{2\pi}\left(\int_0^{2\pi} U\,d\psi\right)\frac{(1-e^2)^{3/2}}{(1+e\cos\nu)^2}\,d\nu. \tag{2.2}$$

Here we have used the fact that

$$\frac{\partial t}{\partial \nu} = \frac{(1-e^2)^{3/2}}{(1+e\cos\nu)^2\omega_0}, \quad \omega_0 = \frac{2\pi}{T_0}, \quad \omega_0 \approx 1, \quad T_0 = 2\pi.$$

In (2.2), $\bar{U}$ is the averaged function. Thus the time averaging of functions depending on ν is reduced to averaging over ν as follows:

$$M_t\{f(\nu)\} = \frac{1}{T_0}\int_0^{T_0} f(\nu)\,dt = \frac{1}{2\pi}\int_0^{2\pi}\frac{(1-e^2)^{3/2} f(\nu)\,d\nu}{(1+e\cos\nu)^2} = (1-e^2)^{3/2} M_\nu\left\{\frac{f(\nu)}{(1+e\cos\nu)^2}\right\}. \tag{2.3}$$

Next, using the expressions for the direction cosines α_2, γ_2, α_3, and γ_3 via the angles ρ, σ, θ, ψ, we obtain the value of the force function U_g averaged over ψ:

$$\langle U_g\rangle_\psi = \frac{3\omega_0^2}{8(1-e^2)^3}(1+e\cos\nu)^3\{(A-C)[(3\cos^2\theta-1)\sin^2\rho(1+\cos 2(\nu-\sigma))+2\sin^2\theta] \\ +(A-B)[2(1-\sin^2\theta\cos^2\varphi)+\sin^2\rho(3\sin^2\theta\cos^2\varphi-1)(1+\cos 2(\nu-\sigma))]\}. \quad (2.4)$$

After averaging over ν, using (2.3), we obtain

$$\tilde{U}_g = \frac{3\omega_0^2}{8(1-e^2)^{3/2}}\{(A-C)[2\sin^2\theta+(3\cos^2\theta-1)\sin^2\rho]+(A-B)[2(1-\sin^2\theta\cos^2\varphi)+(3\sin^2\theta\cos^2\varphi-1)\sin^2\rho]\}. \quad (2.5)$$

Calculating the partial derivatives of the averaged function $\tilde{U}_g$, we obtain

$$\begin{aligned} \frac{\partial\tilde{U}_g}{\partial\psi} &= \frac{\partial\tilde{U}_g}{\partial\sigma} = 0, \\ \frac{\partial\tilde{U}_g}{\partial\theta} &= \frac{3\omega_0}{4(1-e^2)^{3/2}}\sin\theta\cos\theta(2-3\sin^2\rho)[-(A-B)\cos^2\varphi+(A-C)], \\ \frac{\partial\tilde{U}_g}{\partial\rho} &= \frac{3\omega_0^2}{4(1-e^2)^{3/2}}\sin\rho\cos\rho[(A-B)(-1+3\sin^2\theta\cos^2\varphi)+(A-C)(-1+3\cos^2\theta)], \\ \frac{\partial\tilde{U}_g}{\partial\varphi} &= \frac{3\omega_0^2}{4(1-e^2)^{3/2}}\sin\varphi\cos\varphi\sin^2\theta(A-B)(2-3\sin^2\rho). \end{aligned} \quad (2.6)$$

We average the function U_c in a similar way. In the notation introduced in [9], we have

$$\begin{aligned} &\cos^{n+1}\varepsilon_s = (d+g\cos\nu)^{n+1}, \\ &d = \cos\theta\sin\rho\cos(\sigma-\nu), \quad \nu = \psi-\chi, \quad g = \{\sin^2\theta[\sin^2(\sigma-\nu)\sin^2\rho+\cos^2\rho]\}^{1/2}, \\ &\cos\chi = \sin\theta\sin(\sigma-\nu)\{\sin^2\theta[\sin^2(\sigma-\nu)\sin^2\rho+\cos^2\rho]\}^{-1/2} \\ &\sin\chi = \sin\theta\cos\rho\cos(\sigma-\nu)\{\sin^2\theta[\sin^2(\sigma-\nu)\sin^2\rho+\cos^2\rho]\}^{-1/2} \end{aligned} \quad (2.7)$$

By applying the binomial formula, we rewrite the right-hand side of (2.7) as

$$(d+g\cos\nu)^{n+1} = \sum_{k=0}^{n+1} C_{n+1}^k\cos^k\nu(g^k d^{n+1-k}). \quad (2.8)$$

Next, using the expression of the direction cosines α_3, γ_3 of the axis Oz in the frame $OXYZ$ in terms of ρ, σ, θ, ψ [1], we obtain the average over ψ of the force function U_c:

$$\langle U_c\rangle_\psi = -\frac{a_n R_0^2}{(n+1)R^2}\sum_{m=0}^{E(\frac{n+1}{2})} C_{n+1}^{2m} g^m d^{n+1-2m}\frac{(2m-1)!!}{(2m)!!}. \quad (2.9)$$

Here $E(z)$ is the integral part of a number z. In the derivation of (2.9), we have taken into account the fact that

$$\frac{1}{2\pi}\int_0^{2\pi}(d+g\cos\nu)^{n+1}\,d\nu = \sum_{k=0}^{n+1} C_{n+1}^k g^k d^{n+1-k} I_k,$$

$$I_k = \frac{1}{2\pi}\int_0^{2\pi}\cos^k\nu\,d\nu, \quad I_{2m-1} = 0, \quad I_{2m} = \frac{(2m-1)!!}{(2m)!!}.$$

We now average over ν. We denote $u = \sigma - \nu$; then $d = h\cos u$, where $h = \cos\theta\sin\rho$. The expression for g^{2m} in (2.9) can be written as

$$g^{2m} = \left\{\sin^2\theta[\sin^2(\sigma-\nu)\sin^2\rho + \cos^2\rho]\right\}^m = (b + q\sin^2 u)^m,$$
$$b = \sin^2\theta\cos^2\rho, \quad q = \sin^2\theta\sin^2\rho.$$

Again applying the binomial formula, we obtain

$$(b + q\sin^2 u)^m = \sum_{k=0}^{m} C_k^m (q^k b^{m-k})\sin^{2k} u.$$

Thus, as a result of the change, the second averaging of the force function $u = \sigma - \nu$ will be performed over u. To this end, one has to consider the integral

$$\frac{1}{2\pi}\int_0^{2\pi}(b + q\sin^2 u)^m (h\cos u)^{n-2m+1}\,du = \sum_{k=0}^{m} C_k^m (q^k b^{m-k})\frac{1}{2\pi}\int_0^{2\pi}\sin^{2k} u (h\cos u)^{n-2m+1}\,du$$
$$= \sum_{k=0}^{m} h^{n-2m+1} C_k^m (q^k b^{m-k})\frac{1}{2\pi}\int_0^{2\pi}\sin^{2k} u(\cos u)^{n-2m+1}\,du.$$

The integral can be calculated in closed form [9, 11]:

$$\frac{1}{2\pi}\int_0^{2\pi}\sin^{2k} u(\cos u)^{n-2m+1}\,du = \begin{cases} 0, & n = 2l, \\ \dfrac{(2k-1)!![2(k+1-m)-1]!!}{[2(k+l+1-m)]!!}, & n = 2l+1,\ l \in Z \end{cases}$$

Then, after averaging (2.9) over u, we obtain

$$\tilde{U}_{2l+1} = -\delta_l \sum_{m=0}^{l+1}\sum_{k=0}^{m} A_{lmk}(\cos\theta)^{2(l+1-m)}\sin^{2m}\theta(\sin\rho)^{2(l+1-m+k)}(\cos\rho)^{2(m-k)},$$
$$\delta_l = \frac{a_{2l+1}R_0^2(1-e^2)^{3/2}}{2(l+1)P^2},$$
$$A_{lmk} = C_{2(l+1)}^{2m} C_m^k \frac{(2m-1)!!(2k-1)!![2(l+1-m)-1]!!}{(2m)!![2(k+l+1-m)]!!}. \tag{2.10}$$

The force function for the light pressure torque coefficient of the form (1.3) can be written as

$$\tilde{U}_c(\theta,\rho) = \sum_{l=0}^{Q} U_{2l+1}(\theta,\rho), \quad Q = E\Big(\frac{N-1}{2}\Big). \tag{2.11}$$

Calculating the partial derivatives of the function (2.11) with (2.10) taken into account, we obtain

$$\frac{\partial\tilde{U}_c}{\partial\sigma} = \frac{\partial\tilde{U}_c}{\partial\psi} = \frac{\partial\tilde{U}_c}{\partial\varphi} = 0,$$
$$\frac{\partial\tilde{U}_c}{\partial\rho} = -2\sum_{l=0}^{Q}\sum_{m=0}^{l+1}\sum_{k=0}^{m}\delta_l A_{lmk}(\cos\theta)^{2(l+1-m)}(\sin\theta)^{2m}(\sin\rho)^{2(l-m+k)+1}(\cos\rho)^{2(m-k)-1}[(l+1)\cos^2\rho + k - m], \tag{2.12}$$
$$\frac{\partial\tilde{U}_c}{\partial\theta} = -2\sum_{l=0}^{Q}\sum_{m=0}^{l+1}\sum_{k=0}^{m}\delta_l A_{lmk}(\sin\rho)^{2(l+1-m+k)}(\cos\rho)^{2(m-k)}(\sin\rho)^{2l-2m+1}(\cos\theta)^{2l-2m+1}(\sin\theta)^{2m-1}$$
$$\times[m - (l+1)\sin^2\theta].$$

The coefficients δ_l and A_{lmk} are defined in (2.10). We note that the coefficients a_{2l} in the expansion (1.3) disappear under averaging.

Thus, the averaged system of the first approximation becomes

$$\dot{\rho} = 0, \quad \dot{L} = 0, \quad \dot{\sigma} = (L_0 \sin\rho_0)^{-1} \frac{\partial U}{\partial \rho},$$

$$\dot{\theta} = L_0 \sin\theta \sin\varphi \cos\varphi (A^{-1} - B^{-1}) + (L_0)^{-1} \frac{3\omega_0^2}{2(1-e^2)^{3/2}} \left(1 - \frac{3}{2}\sin^2\rho_0\right)(B - A)\sin^2\theta \sin\varphi\cos\varphi,$$

$$\dot{\varphi} = L_0 \cos\theta (C^{-1} - A^{-1}\sin^2\varphi - B^{-1}\cos^2\varphi)$$

$$- 2(L_0 \sin\theta)^{-1} \sum_{l=0}^{Q} \sum_{m=0}^{l+1} \sum_{k=0}^{m} \delta_l A_{lmk} (\sin\rho_0)^{2(l+1-m+k)} (\cos\rho_0)^{2(m-k)} (\cos\theta)^{2l-2m+1} (\sin\theta)^{2m-1} [m - (l+1)\sin^2\theta] \tag{2.13}$$

$$+ \frac{3\omega_0^2}{2(1-e^2)^{3/2}} L_0^{-1} \left(1 - \frac{3}{2}\sin^2\rho_0\right) \cos\theta [(A - C) + (B - A)\cos^2\varphi],$$

$$\delta_l = \frac{a_{2l+1} R_0^2 (1-e^2)^{3/2}}{2(l+1)P^2},$$

where L_0 and ρ_0 are the values of L and ρ at the initial time. Let us study system (2.13). The angular momentum vector remains constant in absolute value and constantly inclined with respect to the normal to the orbit plane. Consider the equations for the nutation angle θ and the proper rotation angle φ (2.13). They describe the motion of the angular momentum vector **L** with respect to the body-fixed frame.

3. EVOLUTION OF THE NUTATION ANGLE AND THE PROPER ROTATION ANGLE

Note that

$$B - A \sim J_0^2 \frac{B - A}{AB}, \quad A - C \sim J_0^2 \frac{A - C}{AC}. \tag{3.1}$$

We substitute the expression (3.1) into (2.13) and denote

$$D_0 = \frac{3\omega_0^2}{2\omega^2(1-e^2)^{3/2}} \left(1 - \frac{3}{2}\sin^2\rho_0\right), \quad \omega = \frac{L_0}{J_0},$$

where ω is the angular precession velocity.

Passing in (2.13) to the slow time $\tau = L_0\beta t$, where $\beta = A^{-1} - B^{-1}$, we obtain

$$\theta' = \sin\theta \sin\varphi \cos\varphi (1 + D_0),$$

$$\varphi' = \cos\theta(\mu - \sin^2\varphi)(1 + D_0) - 2\beta^{-1} L_0^{-2} (\sin\theta)^{-1}$$

$$\times \sum_{l=0}^{Q} \sum_{m=0}^{l+1} \sum_{k=0}^{m} \delta_l A_{lmk} (\sin\rho)^{2(l+1-m+k)} (\cos\rho)^{2m-2k} (\cos\theta)^{2l-2m+1} (\sin\theta)^{2m-1} [m - (l+1)\sin^2\theta], \tag{3.2}$$

$$\mu = -\frac{\gamma}{\beta}, \quad \gamma = B^{-1} - C^{-1}, \quad \beta = A^{-1} - B^{-1}, \quad (\ldots)' = \frac{d}{d\tau}.$$

Taking into account the assumption that $a_{2l+1} \sim \varepsilon$ $(l = 0, \ldots, Q)$ and relations (3.1), we see that β, γ, $\delta_l \sim \varepsilon$. For system (3.2), one has the first integral

$$c = \sin^2\theta(\mu - \sin^2\varphi) - 2\beta^{-1} L_0^{-2} (1 + D_0)^{-1}$$

$$\times \sum_{l=0}^{Q} \sum_{m=0}^{l+1} \sum_{k=0}^{m} \delta_l A_{lmk} (\cos\theta)^{2(l-m+1)} (\sin\theta)^{2m} (\cos\rho_0)^{2(m-k)} (\sin\rho_0)^{2(l+1-m+k)} = \text{const}. \tag{3.3}$$

If the influence of the light pressure torque is absent, i.e., if $a_{2l+1} = 0$ $(l = 0, \ldots, Q)$, then system (3.3) has the form

$$\theta' = \sin\theta \sin\varphi \cos\varphi (1 + D_0),$$

$$\varphi' = \cos\theta(\mu - \sin^2\varphi)(1 + D_0) \tag{3.4}$$

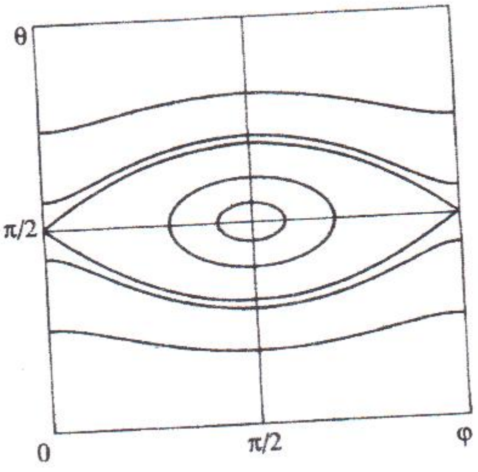

Fig. 1

and has the first integral

$$c_1 = \sin^2\theta(\mu - \sin^2\varphi) = \text{const}. \tag{3.5}$$

If $n = 1$ $(l = 0)$ and $a_c = a_{0c} + a_{1c}\cos\varepsilon_s$, then, according to (3.2), the equations for θ and φ can be written in the slow time as

$$\begin{aligned} &\theta' = (1 + D_0)\sin\theta\sin\varphi\cos\varphi, \\ &\varphi' = (1 + D_0)\cos\theta(\mu_1 - \sin^2\varphi), \\ &\mu_1 = \frac{\alpha - \gamma}{\beta}, \quad \alpha = -\frac{1}{2}L_0^{-2}(1 + D_0)^{-1}(1 - e^2)^{3/2}a_{1c}R_0^2P^{-2}\left(1 - \frac{3}{2}\sin^2\rho_0\right). \end{aligned} \tag{3.6}$$

The paper [2] was the first to propose using the averaging procedure for studying the motion of a nearly dynamically spherical satellite under the action of gravitational torques. In [7], the evolution of rotations of a satellite with close values of three principal moments of inertia under the action of the light pressure torque was studied. Comparing system (3.6) with the corresponding system [10] for the case $a_c = a_{0c} + a_{1c}\cos\varepsilon_s$, we note that the influence of the gravitational torque manifests itself in the factors $1 + D_0$ in the differential equations for θ and φ in the slow time. The expression for μ_1 coincides, up to the factor $\frac{1}{2}(1 + D_0)^{-1}$, with a similar expression in [10]. The existence of this factor is stipulated by the influence of the gravitational forces. In our case, the first integral (3.3) can be written as

$$c_2 = \sin^2\theta(\mu_1 - \sin^2\varphi) = \text{const}, \tag{3.7}$$

where μ_1 is expressed as in (3.5). The first integral (3.7) coincides with the corresponding expression in [10].

The study of system (3.5) is similar to that of the corresponding system in [10]. Here the variables range in the intervals $0 \le \theta < \pi$ and $0 \le \varphi < 2\pi$, and the parameter μ can take various values in $-\infty < \mu < +\infty$. The domain of admissible values of the parameters (c_2, μ_1) can be represented as $D = D_1 \cup D_2 \cup D_3$, where each subdomain is characterized by the following properties: D_1 is determined by the inequalities $\mu_1 \ge c_2 \ge 0$ $(\mu_1 \ge 1)$, the subdomain D_2 is determined by the inequalities $\mu_1 \ge c_2 \ge \mu_1 - 1$ $(0 \le \mu_1 \le 1)$, and the subdomain D_3 is determined by the inequalities $0 \ge c_2 \ge \mu_1 - 1$ $(\mu_1 \le 0)$. The boundaries of the subdomains D_1, D_2, and D_3 are the singular subdomains of system (3.5). In the domains D_1 and D_3, the motion occurs as oscillations in θ and oscillations or rotations in φ. In the domain D_2, oscillations occur in θ and in φ.

We consider eleven different special cases of the choice of the parameter μ_1 (see [10]). For example, for $\mu_1 = 1.7$, the curves of θ against φ obtained numerically from the first integral (3.5) are shown in Fig. 1. These graphs correspond only to oscillations in θ. In the variable φ, oscillations occur only in the interior of the separatrix $\sin^2\theta = \mu_1(\mu_1 - \sin^2\varphi)^{-1}$, and rotations occur in the exterior of the separatrix.

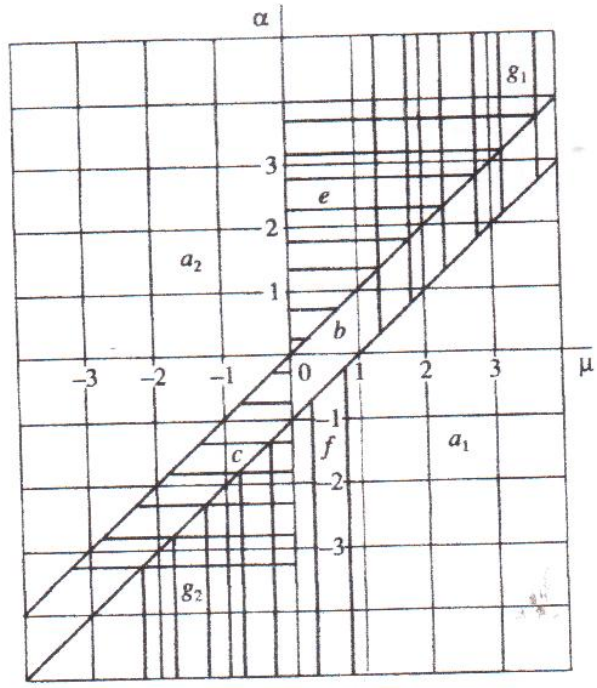

Fig. 2

4. A SPECIAL CASE OF THE EXPRESSION FOR THE SHAPE COEFFICIENT

Let

$$a_c(\cos \varepsilon_s) = \sum_{k=0}^{Q} a_{2k} \cos^{2k} \varepsilon_s + a_3 \cos^3 \varepsilon_s. \tag{4.1}$$

In this case, the equations for θ and φ become

$$\begin{aligned}
&\theta' = (1 + D_0) \sin\theta \sin\varphi \cos\varphi, \\
&\varphi' = (1 + D_0) \cos\theta(\mu_3 - \sin^2\varphi - \alpha_2 \sin^2\theta), \\
&\mu_3 = -\frac{\gamma}{\beta} - \alpha_1 \beta s, \qquad \alpha_1 = (1 + D_0)^{-1} \frac{3a_3 R_0^2}{64 L_0^2 P^2} (1 - e^2)^{3/2} (8 - 40 \sin^2\rho_0 + 35 \sin^4\rho_0), \\
&s = \frac{4 \sin^2\rho_0 (4 - 5 \sin^2\rho_0)}{8 - 40 \sin^2\rho_0 + 35 \sin^4\rho_0}.
\end{aligned} \tag{4.2}$$

We note that the terms containing even powers of a_{2k} disappear after averaging.

Taking into account the assumption that $a_3 \sim \varepsilon$ and relation (1.9), we find that β, α, and γ are quantities of the order of ε.

For system (4.2), we have the first integral

$$c_2 = \sin^2\theta \left(\mu_2 - \sin^2\varphi - \frac{1}{2}\alpha_2 \sin^2\theta\right) = \text{const}. \tag{4.3}$$

In [9], the motion of a triaxial nearly spherical satellite under the action of the light pressure torque was studied, where the light pressure torque coefficient has the form (4.1) and the coefficient a_c satisfies the same assumptions as in our problem.

Comparing system (4.2) for θ and φ with the corresponding system in [9], we note that the influence of the gravitational torque manifests itself in the factor $(1 + D_0)$. The first integrals of the systems compared also coincide up to the factor $(1 + D_0)^{-1}$.

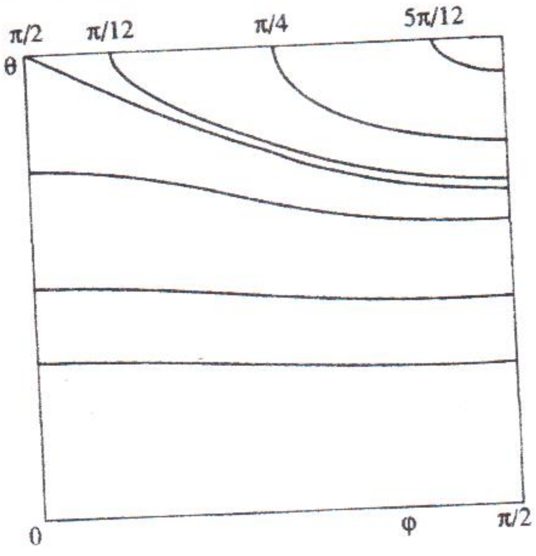

Fig. 3

In system (4.2), the variables θ and φ range in the intervals $0 \le \theta < \pi$ and $0 \le \varphi < \pi$. The parameter μ can take arbitrary values, $-\infty < \mu < +\infty$. The domain of admissible values (α, μ) is shown in Fig. 2.

We consider nine different typical cases of the choice of the parameters (μ, α) corresponding to each of the domains shown in Fig. 2 (see [9]). For example, the family of phase trajectories of the averaged system in the plane θ, φ for $\mu = -5$ and $\alpha = -2$ (case (a_2)) is shown in Fig. 3. These graphs correspond to oscillations in the angle θ, and in the angle φ, we have either oscillations (in the interior of the separatrix) or rotations (in the exterior of the separatrix); the stationary points $(\frac{1}{2}\pi, \frac{1}{2}\pi)$ are of the center type and the stationary points $(\frac{1}{2}\pi, 0)$ are of the saddle type. In case (a_1), we also have a similar character of the dependence of θ on φ. The other domains of the dependence (μ, α) can be studied by using considerations similar to those in [9].

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