MOTION OF A SATELLITE RELATIVE TO THE CENTER OF MASS UNDER THE ACTION OF MOMENTS OF LIGHT-PRESSURE FORCES

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There have been many studies of the motion of a satellite relative to the center of mass under the action of moments of various kinds of forces (gravitational, magnetic, light-pressure, and others; see [1-4] and the literature cited there). Estimation of the moments of the perturbing forces [1] indicates that at heights greater than 35,000-40,000 km above the Earth's surface, moments of light-pressure forces exert the greatest effect on spacecraft. Using the averaging method, we investigate the motion relative to the center of mass, caused by moments of light-pressure forces, for a spacecraft with a triaxial ellipsoid of inertia in the case in which the spacecraft is a body of revolution.

1. Consider the motion of a spacecraft relative to the center of mass under the action of moments of light-pressure forces.

We introduce three right Cartesian coordinate systems, the coordinate origin coinciding with the center of inertia of the satellite [1, 2]. The OXYZ system moves translationally around the orbit together with the satellite; the Y axis is parallel to the normal to the orbital plane; the Z axis is parallel to the direction of the radius vector of the orbit and its perigee; and the X axis is parallel to the velocity vector of the center of mass of the satellite at the perigee. The position of the kinetic-moment vector **L** in the OXYZ system is specified by the angles ρ and σ, as shown in Fig. 1a.

To construct the OL_1L_2L coordinate system associated with vector **L**, we run the L_1 axis in the OYL plane perpendicularly to vector **L**, such that it makes an obtuse angle with the Y axis. The L_2 axis completes the L_1 and **L** axes to right coordinate systems.

The axes of the associated Oxyz system coincide with the principal central axes of inertia of the satellite.

We specify the mutual disposition of the principal central axes of inertia and the L, L_1, L_2 axes by the Euler angles φ, ψ, θ (Fig. 1b).

In this case the direction cosines of the Ox, Oy, Oz axes relative to the OL_1L_2L system can be expressed in terms of the Euler angles φ, ψ, θ via the formulas

$$\begin{gathered}\alpha_{11}=\cos\psi\cos\varphi-\cos\theta\sin\psi\sin\varphi,\quad \alpha_{12}=-\cos\psi\sin\varphi-\cos\theta\sin\psi\cos\varphi\\ \alpha_{13}=\sin\theta\sin\psi,\quad \alpha_{21}=\sin\psi\cos\varphi+\cos\theta\cos\psi\sin\varphi\\ \alpha_{22}=-\sin\psi\sin\varphi+\cos\theta\cos\psi\cos\varphi,\quad \alpha_{23}=-\sin\theta\cos\psi\\ \alpha_{31}=\sin\theta\sin\varphi,\quad \alpha_{32}=\sin\theta\cos\varphi,\quad \alpha_{33}=\cos\theta\end{gathered} \tag{1.1}$$

We will assume that the spacecraft moves in an elliptical orbit around the Sun; we will disregard moments of all forces except for light-pressure forces. We will further assume that the surface of the spacecraft is a surface of revolution, with the unit vector of the axis of symmetry **k** directed along the Oz axis. As shown in [1, 3], in this case we have the following formula for the moment of light-pressure forces **M** acting on the satellite:

$$\mathbf{M}=(a_0(\varepsilon_s)R_0^2/R^2)\mathbf{e}_r\times\mathbf{k} \tag{1.2}$$

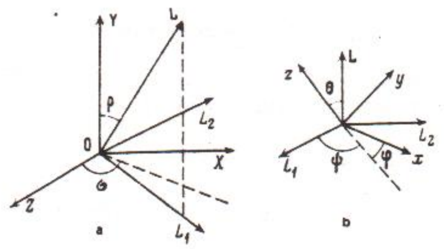

Fig. 1

In the case of complete absorption,

$$a_c(\varepsilon_s)\frac{R_0^2}{R^2}=p_0 S(\varepsilon_s)z_0'(\varepsilon_s),\qquad p_0=\frac{E_0}{c}\left(\frac{R_0}{R}\right)^2 \tag{1.3}$$

In (1.2) and (1.3), $\mathbf{e}_r$ is the unit vector in the direction of the radius vector of the orbit; ε_s is the angle between the directions of $\mathbf{e}_r$ and $\mathbf{k}$, so that $|\mathbf{e}_r\times\mathbf{k}|=\sin\varepsilon_s$; R is the instantaneous distance from the center of the Sun to the center of mass of the satellite; R_0 is a fixed value of R, e.g., at the initial instant; $a_c(\varepsilon_s)$ is the coefficient of the moment of light-pressure forces; S is the area of the "shadow" on the plane normal to the flux; z_0' is the distance from the center of mass to the center of pressure; c is the speed of light; and E_0 is the energy flux of light pressure at a distance R from the center of the Sun. If R_0 is the radius of the Earth's orbit, then E_0 = 1200 kcal·m^{-2}·hr, $p_0= E_0/c=4{,}72\cdot10^{-6}$ hr·m^{-2}.

Following [1], we assume that $a_c=a_c(\cos\varepsilon_s)$; we approximate a_c by polynomials in degrees of cos ε_s.

The moments of light-pressure forces have a force function that depends only on the position of the axis of symmetry of the body in space [1]. We expand $a_c(\cos\varepsilon_s)$ in a Taylor series:

$$a_c=a_{0c}+a_{1c}\cos\varepsilon_s+\dots \tag{1.4}$$

Now we consider only the first two terms of the expansion.

The equations of perturbed motion of the satellite with a force function in the variables $L,\rho,\sigma,\varphi,\psi,\theta$ have the form [2, 4]:

$$\sigma^{\cdot}=\frac{1}{L\sin\rho}\frac{\partial U}{\partial\rho},\quad \rho^{\cdot}=-\frac{1}{L\sin\rho}\frac{\partial U}{\partial\sigma}+\frac{\mathrm{ctg}\,\rho}{L}\frac{\partial U}{\partial\psi}$$
$$L^{\cdot}=\frac{\partial U}{\partial\psi},\quad \theta^{\cdot}=L\sin\theta\sin\varphi\cos\varphi\left(\frac{1}{A}-\frac{1}{B}\right)-\frac{1}{L\sin\theta}\frac{\partial U}{\partial\varphi}+ \tag{1.5}$$
$$+\frac{\mathrm{ctg}\,\theta}{L}\frac{\partial U}{\partial\psi},\quad \varphi^{\cdot}=L\sin\theta\cos\theta\left(\frac{1}{C}-\frac{\sin^2\varphi}{A}-\frac{\cos^2\varphi}{B}\right)+$$
$$+\frac{1}{L\sin\theta}\frac{\partial U}{\partial\theta},\quad \psi^{\cdot}=L\left(\frac{\sin^2\varphi}{A}+\frac{\cos^2\varphi}{B}\right)-\frac{1}{L}\left(\frac{\partial U}{\partial\rho}\mathrm{ctg}\,\rho+\frac{\partial U}{\partial\theta}\mathrm{ctg}\,\theta\right)$$

In some cases, it is convenient to employ the kinetic energy as a variable instead of the angle θ:

$$T={}^1/_2L^2[(\sin^2\varphi/A+\cos^2\varphi/B)\sin^2\theta+\cos^2\theta/C] \tag{1.6}$$

whose derivatives has the form

$$T^{\cdot}=\frac{2T}{L}M_3+L\sin\theta\left[\cos\theta\left(\frac{\sin^2\varphi}{A}+\frac{\cos^2\varphi}{B}-\frac{1}{C}\right)(M_2\cos\psi-\right. \tag{1.7}$$

$$-\dot{M}_1 \sin\psi) + \sin\varphi\cos\varphi\,(1/A - 1/B)\,(M_1\cos\psi + M_2\sin\psi)\Big] \qquad (1.7)$$

Force function U depends on the time t via the true anomaly $\nu(t)$, and on the direction cosines $\alpha_3, \beta_3, \gamma_3$ of the Oz axis relative to the OXYZ system:

$$U = U(\nu(t), \alpha_3, \beta_3, \gamma_3) \qquad (1.8)$$

The projections of the vector of the moment of forces M onto the axes of the OXYZ system have the form [1]:

$$M_X = \frac{\partial U}{\partial \gamma_3}\beta_3 - \frac{\partial U}{\partial \beta_3}\gamma_3, \quad M_Y = \frac{\partial U}{\partial \alpha_3}\gamma_3 - \frac{\partial U}{\partial \gamma_3}\alpha_3, \quad M_Z = \frac{\partial U}{\partial \beta_3}\alpha_3 - \frac{\partial U}{\partial \alpha_3}\beta_3 \qquad (1.9)$$

while the projections onto the OLL_1L_2 axes can be expressed in terms of M_X, M_Y, M_Z via the formulas

$$\begin{aligned} M_1 &= (M_X \sin\sigma + M_Z\cos\sigma)\cos\rho - M_Y \sin\rho \\ M_2 &= M_X\cos\sigma - M_Z\sin\sigma \\ M_3 &= (M_X\sin\sigma + M_Z\cos\sigma)\sin\rho + M_Y\cos\rho \end{aligned} \qquad (1.10)$$

To system (1.5) we need to add an equation that describes the change in the true anomaly over time:

$$\frac{d\nu}{dt} = \frac{\omega_0}{(1-e^2)^{3/2}}(1+e\cos\nu)^2, \quad \omega_0 = \frac{2\pi}{N_0} = \left[\frac{\mu(1-e^2)^3}{P^3}\right]^{1/2} \qquad (1.11)$$

ω_0 is the average angular velocity of motion of the center of mass along the elliptical orbit; N_0 is the orbital period of the satellite; e and P are the eccentricity and focal parameter of the orbit; and μ is the product of the universal gravitational constant and the mass of the Sun.

Moment of forces (1.2) corresponds to force function $U(\cos\varepsilon_s) = -R_0^2/R^2 \int a_e(\cos\varepsilon_s)\,d(\cos\varepsilon_s)$.

Let us consider the two cases $a_e(\cos\varepsilon_s) = a_{0e}$ and $a_e(\cos\varepsilon_s) = a_{1e}\cos\varepsilon_s$, which correspond to the first two terms of the expansion of $a_e(\cos\varepsilon_s)$ in a Taylcr series (1.4).

In these cases, the force functions have the form $U(\cos\varepsilon_s) = -R_0^2/R^{-2}a_{0e}\cos\varepsilon_s$ and $U(\cos\varepsilon_s) = -R_0^2/R^{-2}a_{1e}\cos^2\varepsilon_s$ respectively, where $\cos\varepsilon_s = \gamma_3\cos\nu + \alpha_3\sin\nu$.

Note that the first case corresponds, e.g., to a spherical satellite whose center of mass is displaced relative to the center of the sphere.

We introduce small parameters into system (1.5), (1.11). Let us assume that $\omega_0 \sim \varepsilon$, and also that $a_{0e} \sim \varepsilon \ll 1$ or $a_{1e} \sim \varepsilon$, depending on which case is considered. Let us investigate the solution of system (1.5), (1.11) for small ε on a large time interval $t \sim \varepsilon^{-1}$. We employ the averaging method [5, 6] to solve the problem. The error of the averaged solution for the slow variables is of order ε on the time interval over which the body executes $\sim\varepsilon^{-1}$ revolutions. We perform averaging with respect to Euler-Poinsot motion using the procedure of [2, 4] for nonresonant cases.

2. Let us consider unperturbed motion (ε = 0) when the moment of the light-pressure forces (1.2) is equal to zero. In this case the rotation of the satellite is Euler-Poinsot motion. The quantities σ, ρ, L, T, ν become constants, while θ, φ, ψ become certain functions of time. Also, σ, ρ, L, T, ν, will be slow variables in perturbed motion, while the Euler angles θ, φ, ψ will be fast variables.

Let us average the first three equations of system (1.5), (1.7) along the trajectory of unperturbed motion. According to [2], averaging is performed first with respect to ψ, then with respect to θ and φ, which are related by (1.6). It is performed along closed trajectories of the kinetic-moment vector in Euler-Poinsot motion. Averaging of the right sides of the first three equations of (1.5) with respect to ψ leads to the equations

$$\dot{\sigma} = \frac{1}{L\sin\rho}\frac{\partial\langle U\rangle_\psi}{\partial\rho}, \quad \dot{\rho} = -\frac{1}{L\sin\rho}\frac{\partial\langle U\rangle_\psi}{\partial\sigma}, \quad \dot{L} = 0, \quad \langle U\rangle_\psi = \frac{1}{2\pi}\int_0^{2\pi} U\,d\psi \qquad (2.1)$$

Let us average the right side of (1.7) with respect to the angle ψ. In the first case we have, on the basis of (1.9) and (1.10),

$$M_1=-R_0^2R^{-2}a_{0e}[\alpha_3 \sin\rho\cos\nu+\beta_3\sin(\sigma-\nu)\cos\rho-\gamma_3\sin\rho\sin\nu]$$
$$M_2=-R_0^2R^{-2}a_{0e}\beta_3\cos(\sigma-\nu) \quad (2.2)$$
$$M_3=-R_0^2R^{-2}a_{0e}[-\alpha_3\cos\rho\cos\nu+\beta_3\sin(\sigma-\nu)\sin\rho+\gamma_3\cos\rho\sin\nu]$$

Using the expressions for the direction cosines $\alpha_3, \beta_3, \gamma_3$ of the Oz axis relative to the OXYZ system [1]:

$$\alpha_3=\sin\psi\sin\theta\cos\rho\sin\sigma-\cos\psi\sin\theta\cos\sigma+\cos\theta\sin\rho\sin\sigma$$
$$\beta_3=-\sin\psi\sin\theta\sin\rho+\cos\theta\cos\rho \quad (2.3)$$
$$\gamma_3=-\sin\psi\sin\theta\cos\rho\cos\sigma+\cos\psi\sin\theta\sin\sigma+\cos\theta\sin\rho\cos\sigma$$

we obtain from (2.2), using elementary manipulations,

$$\langle M_3\rangle_\psi=0, \quad \langle M_2\cos\psi-M_1\sin\psi\rangle_\psi=0$$
$$\langle M_1\cos\psi+M_2\sin\psi\rangle_\psi=R_0^2R^{-2}a_{0e}\sin\theta\sin\rho\cos(\sigma-\nu)$$

It follows that, after averaging with respect to ψ, Eq. (1.7) becomes

$$T^*=\frac{R_0^2}{R^2}a_{0e}\left(\frac{1}{A}-\frac{1}{B}\right)L\sin^2\theta\sin\varphi\cos\varphi\sin\rho\cos(\sigma-\nu) \quad (2.4)$$

Projection of vector **L** onto the axes of the associated coordinate system yields

$$L_x=L\sin\theta\sin\varphi, \quad L_y=L\sin\theta\cos\varphi, \quad L_z=L\cos\theta \quad (2.5)$$

As a result we have

$$T^*=\frac{R_0^2}{R^2}a_{0e}\left(\frac{1}{A}-\frac{1}{B}\right)\frac{L_xL_y}{L}\sin\rho\cos(\sigma-\nu) \quad (2.6)$$

Now we average the right side of (1.7) with respect to ψ for the second case. The formula for the moment (1.2) of the light-pressure forces yields, with allowance for the fact that the Oz axis coincides with the axis of symmetry of the satellite,

$$\mathbf{M}=a_{1e}R_0^2R^{-2}(-\gamma_3\beta_3\mathbf{Ox}+\alpha_3\gamma_3\mathbf{Oy}) \quad (2.7)$$

As we know [2], the moment of gravitational forces acting on the satellite has the form

$$\mathbf{M}=3\mu R^{-3}[(C'-B')\gamma_3\beta_3\mathbf{Ox}+(A'-C')\alpha_3\gamma_3\mathbf{Oy}+(B'-A')\alpha_3\beta_3\mathbf{Oz}] \quad (2.8)$$

Here A', B', C' are the principal central moments of inertia of a satellite in a gravitational field.

The moment of light-pressure forces as given by (2.7) coincides with the moment of gravitational forces (2.8) acting on a satellite in a gravitational field, whose principal central moments of inertia have the form

$$A'=B'=2a_{1e}R_0^2R/3\mu, \quad C'=a_{1e}R_0^2R/3\mu \quad (2.9)$$

The motion of a satellite relative to the center of mass under the action of gravitational moments was investigated in [2], where the projections of the moment of gravitational forces onto the OL_1, OL_2, OL axes can be represented as follows for a triaxial satellite:

$$M_1=3\omega_0^2(1+e\cos\nu)^3(1-e^2)^{-3}\sum_{j=1}^{3}(\delta_2\delta_j s_{3j}-\delta_3\delta_j s_{2j})$$
$$M_2=3\omega_0^2(1+e\cos\nu)^3(1-e^2)^{-3}\sum_{j=1}^{3}(\delta_3\delta_j s_{1j}-\delta_1\delta_j s_{3j})$$

$$M_3 = 3\omega_0^2(1+e\cos\nu)^3(1-e^2)^{-3}\sum_{j=1}^{3}(\delta_1\delta_j s_{2j} - \delta_2\delta_j s_{1j})$$

Here $s_{ij} = A'\alpha_{i1}\alpha_{j1} + B'\alpha_{i2}\alpha_{j2} + C'\alpha_{i3}\alpha_{j3}$, α_{ij} are defined by formulas (1.1); δ_1, δ_2, δ_3 are the direction cosines of vector $\mathbf{e}_r$ in the OL_1L_2L system, equal to $\delta_1 = \cos\rho\cos(\nu-\sigma)$, $\delta_2 = \sin(\nu-\sigma)$, $\delta_3 = \sin\rho\cos(\nu-\sigma)$.

Calculations show that

$$\langle M_3\rangle_\psi = 0, \quad \langle M_1\cos\psi + M_2\sin\psi\rangle_\psi = \frac{3\omega_0^2(1+e\cos\nu)^3}{(1-e^2)^3}(\delta_1^2+\delta_2^2-2\delta_3^2)\times$$
$$\times\frac{1}{2}\sin\theta\cos\theta(A'\sin^2\varphi + B'\cos^2\varphi - C')$$
$$\langle M_2\cos\psi - M_1\sin\psi\rangle_\psi =$$
$$= \frac{3\omega_0(1+e\cos\nu)^3}{(1-e^2)^3}(2\delta_3^2-\delta_1^2-\delta_2^2)\frac{1}{2}(A'-B')\sin\theta\sin\varphi\cos\varphi$$

Allowing for the fact that A' = B', in accordance with (2.9), we have

$$\langle M_3\rangle_\psi = \langle M_2\cos\psi - M_1\sin\psi\rangle_\psi = 0,$$
$$\langle M_1\cos\psi + M_2\sin\psi\rangle_\psi = \frac{3\omega_0^2(1+e\cos\nu)^3}{2(1-e^2)^3}(\delta_1^2+\delta_2^2-2\delta_3^2)(A'-C')\sin\theta\cos\theta$$

It follows that, after averaging with respect to ψ, Eq. (1.7) becomes

$$T^{\cdot} = L(A'-C')\frac{3\omega_0^2(1+e\cos\nu)^3}{2(1-e^2)^3}\left(\frac{1}{A'}-\frac{1}{B'}\right)\sin^2\theta\cos\theta\sin\varphi\times$$
$$\times\cos\varphi\,(\delta_1^2+\delta_2^2-2\delta_3^2) = (A'-C')\times$$
$$\times\left(\frac{1}{A}-\frac{1}{B}\right)\frac{L_xL_yL_z}{L^2}\frac{3\omega_0^2(1+e\cos\nu)^3}{2(1-e^2)^3}(\delta_1^2+\delta_2^2-2\delta_3^2) \qquad (2.10)$$

In formulas (2.6) and (2.9), only L_x, L_y, L depend on the fast variables. But the average values of the products L_xL_y and $L_xL_yL_z$ over the polhode of unperturbed motion are equal to zero, in view of the fact that the segments of the polhodes are symmetrical with respect to the O_{xy}, O_{xz}, O_{yz} coordinate planes. In both cases, therefore, after averaging over the polhode Eq. (1.7) assumes the form $T^{\cdot} = 0$, and hence $T = T_0$ = const. This means that, averaging, the equations for the angles σ and ρ can be treated independently of the other equations for slow variables.

The average values with respect to ψ of the force function in the first and second cases will be, respectively,

$$\langle U\rangle_\psi = -a_{00}R_0^2R^{-2}\cos\theta\sin\varphi\cos(\sigma-\nu)$$
$$\langle U\rangle_\psi = -a_{10}R_0^2R^{-2}(1-{}^3/_2\sin^2\theta)\sin^2\rho\cos^2(\sigma-\nu) \qquad (2.11)$$

Now it is necessary to average the force function along the polhode of unperturbed motion. Unperturbed motion of a triaxial satellite was investigated in detail in [4]. To calculate the averaged value of the force function in first approximation, we need to determine

$$\lim_{N\to\infty}\frac{1}{N}\int_0^N\cos\theta(t)\,dt, \qquad \lim_{N\to\infty}\frac{1}{N}\int_0^N\sin^2\theta(t)\,dt$$

where $\theta(t)$ is the angle of nutation as a function of time in unperturbed motion.

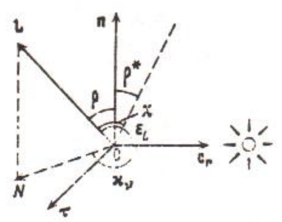

Fig. 2

For the sake of being definite, we take $A > B > C$. Function $\theta(t)$ is defined differently, depending on the sign of $2T_0B - L_0^2$. If $2T_0B = L_0^2 > 0$, then (see [4, 7]) $\cos\theta(\tau) = a\,\mathrm{dn}\,\tau$. For $2T_0B - L_0^2 < 0$, we obtain $\cos\theta(\tau) = a\,\mathrm{cn}\,\tau$. Here $\mathrm{dn}\,\tau$, $\mathrm{cn}\,\tau$ are elliptic functions [8] that are periodic in τ with period $N_\tau = 4K(k^2)$, $K(k^2)$ is a complete elliptic integral of the first kind,

$$a^2 = \frac{\varepsilon_1 + h}{1+\varepsilon_1}, \quad \varepsilon_1 = \frac{C}{A}\,\frac{A-B}{B-C}, \quad h = \left(\frac{2T_0}{L_0^2} - \frac{1}{B}\right)\frac{BC}{B-C},$$

$$\tau = \beta t, \quad \beta = \frac{L_0(A-B)}{AB\varepsilon_1}\sqrt{1+\varepsilon_1}, \quad k^2 = \frac{a^2-h}{a^2}.$$

The formulas for the modulus of the elliptic functions k^2 and β are given for $h > 0$. The remaining expressions are valid for any h.

Motions in the neighborhood of the Oz axis (axis of moment of inertia C) correspond to $h>0$ $(2T_0B-L_0^2>0)$, while motions in the neighborhood of Ox (axis of moment of inertia A) correspond to $h<0$ $(2T_0B-L_0^2<0)$; motion along the separatrix corresponds to $h=0$ $(2T_0B=L_0^2)$.

Using the formulas for integrals of elliptic functions [8], and allowing for the fact that $\cos\theta(\tau) = a\,\mathrm{dn}\,\tau$, $\cos\theta(\tau) = a\,\mathrm{cn}\,\tau$ we obtain

$$\lim_{N\to\infty}\frac{1}{N}\int_0^N \cos\theta(t)\,dt = \frac{1}{N_\tau}\int_0^{N_\tau}\cos\theta(\tau)\,d\tau = \begin{cases} \dfrac{\pi a}{2K(k^2)} & \text{for } 2T_0B-L_0^2>0 \\ 0 & \text{for } 2T_0B-L_0^2<0 \end{cases} \tag{2.12}$$

$$\lim_{N\to\infty}\frac{1}{N}\int_0^N \sin^2\theta(t)\,dt = \frac{1}{N_\tau}\int_0^{N_\tau}\sin^2\theta(\tau)\,d\tau =$$

$$= \begin{cases} 1 - a\dfrac{E(k^2)}{K(k^2)} & \text{for } 2T_0B-L_0^2>0 \\ 1 - \dfrac{a^2}{k^2}\left[k^2 - 1 + \dfrac{E(k^2)}{K(k^2)}\right] & \text{for } 2T_0B-L_0^2<0 \end{cases} \tag{2.13}$$

Here $E(k^2)$ is a complete elliptic integral of the second kind.

As a result of averaging over the polhode in the first case we obtain the following system of equations for ρ and σ:

$$\begin{aligned} \rho^{\cdot} &= -a_{00}R_0^3(L_0R^3)^{-1}F\sin(\sigma-\nu) \\ \sigma^{\cdot} &= -a_{00}R_0^3(L_0R^3)^{-1}F\,\mathrm{ctg}\,\rho\cos(\sigma-\nu) \end{aligned} \tag{2.14}$$

$$F = \frac{\pi a}{2K(k^2)}, \quad 2T_0B-L_0^2>0; \quad F=0, \quad 2T_0B-L_0^2<0$$

In the second case the averaged system of the first approximation for ρ and σ becomes

$$\rho^{\cdot} = -a_{1c}R_0^2(2L_0R^3)^{-1}G\sin\rho\sin 2(\sigma-\nu), \tag{2.15}$$

$$\sigma^{\cdot} = -a_{1c}R_0^2(L_0R^3)^{-1}G\cos\rho\cos^2(\sigma-\nu)$$

$$G = \begin{cases} \frac{1}{2}\left[3a^2\frac{E(k^2)}{K(k^2)}-1\right] & \text{for } 2T_0B-L_0^2>0 \\ \frac{1}{2}\left\{\frac{3a^2}{k^2}\left[k^2-1+\frac{E(k^2)}{K(k^2)}\right]-1\right\} & \text{for } 2T_0B-L_0^2<0 \end{cases}$$

The earlier notation is retained for the slow averaged variables.

In systems (2.14) and (2.15) it is convenient to change over to the new independent variable $\nu = \nu(t)$. In view of (1.11) and the equation of motion of the center of mass of a satellite along a plane elliptic orbit $R = P/(1 + e\cos\nu)$, after changing over from the independent variable t to the variable ν, systems (2.14) and (2.15) become

$$d\rho/d\nu = -a_{0c}R_0^2(L_0\sqrt{\mu P})^{-1}F\sin(\sigma-\nu) \tag{2.16}$$

$$d\sigma/d\nu = -a_{0c}R_0^2(L_0\sqrt{\mu P})^{-1}F\operatorname{ctg}\rho\cos(\sigma-\nu)$$

$$d\rho/d\nu = -a_{1c}R_0^2(2L_0\sqrt{\mu P})^{-1}G\sin\rho\sin 2(\sigma-\nu) \tag{2.17}$$

$$d\sigma/d\nu = -a_{1c}R_0^2(L_0\sqrt{\mu P})^{-1}G\cos\rho\cos^2(\sigma-\nu)$$

We set $\varkappa_\nu = \sigma-\nu$. The coordinate $\varkappa_\nu$ is the angle between the instantaneous radius vector $\mathbf{e}_r$ of the orbit and the projection of vector $\mathbf{L}$ onto the orbital plane. Thus, angles $\rho, \varkappa_\nu$ give the position of vector $\mathbf{L}$ in the rotating coordinate system $\mathbf{n}, \boldsymbol{\tau}, \mathbf{e}_r$, where $\mathbf{n}$ lies along the normal to the orbital plane, $\boldsymbol{\tau}$ along the transversal (Fig. 2). In the variables $\rho, \varkappa_\nu$, Eqs. (2.16) and (2.17) become

$$d\rho/d\nu = -a_{0c}R_0^2(L_0\sqrt{\mu P})^{-1}F\sin\varkappa_\nu \tag{2.18}$$

$$d\varkappa_\nu/d\nu = -a_{0c}R_0^2(L_0\sqrt{\mu P})^{-1}F\operatorname{ctg}\rho\cos\varkappa_\nu-1$$

$$d\rho/d\nu = -a_{1c}R_0^2(2L_0\sqrt{\mu P})^{-1}G\sin\rho\sin 2\varkappa_\nu \tag{2.19}$$

$$d\varkappa_\nu/d\nu = -a_{1c}R_0^2(L_0\sqrt{\mu P})^{-1}G\cos\rho\cos^2\varkappa_\nu-1$$

Systems (2.18) and (2.19) are autonomous and have the first integrals

$$L_0\cos\rho - a_{0c}R_0^2(\sqrt{\mu P})^{-1}F\cos\varepsilon_L = \text{const} \tag{2.20}$$

$$L_0\cos\rho - a_{1c}R_0^2(2\sqrt{\mu P})^{-1}G\cos^2\varepsilon_L = \text{const} \tag{2.21}$$

Here ε_L is the angle between vectors $\mathbf{L}$ and $\mathbf{e}_r$ (Fig. 2): $\cos\varepsilon_L = \sin\rho\cos\varkappa_\nu$.

The first integrals (2.20) and (2.21) differ from the first integrals for systems that describe the variation of the angles ρ and $\varkappa_\nu$ in the case of a dynamically symmetrical satellite [1] only in terms of the multipliers F and G. Therefore, the results of [1] can be carried over to the case under consideration of motion relative to the center of mass of a spacecraft with triaxial ellipsoid of inertia that is a body of revolution.

The first integral in (2.20) can be written in the form

$$\cos\chi = \text{const} \tag{2.22}$$

where χ is the angle between vector $\mathbf{L}$ and straight line OP in the $(\mathbf{n}, \mathbf{e}_r)$ plane, which makes an angle ρ^* with vector $\mathbf{n}$. The angle ρ^* is given by the equation $\operatorname{tg}\rho^* = -n_0$, $n_0 = a_{0c}R_0^2/(L_0\sqrt{\mu P})$.

Equation (2.22) means that in the $(\mathbf{n}, \boldsymbol{\tau}, \mathbf{e}_r)$ coordinate system the motion of the kinetic-moment vector $\mathbf{L}$ is uniform rotation with respect to ν about the straight line OP.

The angular rotational velocity of vector **L** was determined in [1]; it is equal to $-\sqrt{1+n_0^2}$.

Note that if the initial conditions T_0 and L_0 are such that $2T_0B - L_0^2 < 0$, then $F = 0$ and the angle ρ^* is zero. Then the motion of vector **L** in the $(\mathbf{n}, \boldsymbol{\tau}, \mathbf{e}_r)$ system is rotation with angular velocity 1 about the vector **n**. Vector **L** will be constant in the OXYZ system. This means that the osculating elements ρ and σ do not vary.

If, however, T_0 and L_0 are such that $2T_0B - L_0^2 > 0$, then the angle ρ^* is nonzero and vector **L** rotates about the OP axis which is inclined with respect to the light source; the angle of inclination of the body is greater, the greater n_0.

The motion of the satellite can be investigated in the second case on the basis of the first integral in (2.21) in the same way as was done in [1] for the case of a dynamically symmetrical satellite in a gravitational field in a circular orbit, since system (2.19) differs from the system of equations describing the motion of a satellite in a gravitational field in a circular orbit only in terms of the multiplier G, which depends on the initial data T_0 and L_0.

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