PERTURBED ROTATION OF A RIGID BODY RELATIVE TO A FIXED POINT

D. D. Leshchenko and S. N Sallam

Izv. AN SSSR. Mekhanika Tverdogo Tela,
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In this paper, we investigate perturbed rotational motion of a rigid body, that is close to regular precession in the Lagrange case, when the restoring moment depends on the angle of nutation. It is assumed that the angular velocity of the body is large; its direction lies close to the axis of dynamic symmetry of the body; and the two projections of the vector of the perturbing moment onto the principal axes of inertia of the body are small as compared to the restoring moment, while the third is of the same order as it. A small parameter is introduced in special fashion, and the averaging method is employed. The averaged system of equations is obtained in first approximation. Some examples are considered.

1. STATEMENT OF THE PROBLEM

Consider the motion of a dynamically symmetrical rigid body relative to fixed point O, in response to a perturbing moment that depends on the angle of nutation θ and the perturbing moment. The equations of motion have the form

$$
\begin{gathered}
Ap^{\cdot}+(C-A)qr=k(\theta)\sin\theta\cos\varphi+M_1 \\
Aq^{\cdot}+(A-C)pr=-k(\theta)\sin\theta\sin\varphi+M_2 \\
Cr^{\cdot}=M_3,\ M_i=M_i(p,\ q,\ r,\ \psi,\ \theta,\ \varphi,\ t)\quad (i=1,\ 2,\ 3) \\
\psi^{\cdot}=(p\sin\varphi+q\cos\varphi)\operatorname{cosec}\theta,\ \theta^{\cdot}=p\cos\varphi-q\sin\varphi \\
\varphi^{\cdot}=r-(p\sin\varphi+q\cos\varphi)\operatorname{ctg}\theta
\end{gathered}
\tag{1.1}
$$

The dynamic Euler equations are written in the projections onto the principal axes of inertia of the body, passing through point O. Here p, q, r are the projections of the angular velocity vector of the body onto these axes; M_i (i = 1, 2, 3) are the projections of the vector of the perturbing moment onto the same axes, which are 2π-periodic functions of the Euler angles ψ, θ, ϕ; ψ is the angle of precession; θ is the angle of nutation; ϕ is the angle of intrinsic rotation; and A and C are the equatorial and axial moments of inertia of the body relative to point O, $A \neq C$. It is assumed that the body is acted upon by a restoring moment that depends on the angle of nutation $k(\theta)$. In the case of a heavy top, we have $k = mgl$, where m is the mass of the body; g is the acceleration due to gravity; and l is the distance from fixed point O to the center of gravity of the body.

The perturbing moments M_i in (1.1) are assumed to be known functions of their arguments. In the absence of perturbations $M_i = 0$, i = 1, 2, 3 and $k(\theta)$ = const, Eqs. (1.1) correspond to the Lagrange case.

We will make the following initial assumptions:

$$
p^2+q^2\ll r^2,\ Cr^2\gg k,\ |M_i|\ll k\ \ (i=1,\ 2),\ M_3\sim k
\tag{1.2}
$$

which mean that the direction of the angular velocity of the body is close to the axis of dynamic symmetry; the angular velocity is large, so that the kinetic energy of the body is much greater than the potential energy resulting from the restoring moment; and two projections of the vector of the perturbing moment onto the principal axes of inertia of the body are small as compared to the restoring moment, while the third is of the same order as it. On the basis of inequalities (1.2), we introduce the small parameter ϵ, and we set

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$$p=\varepsilon P,\ q=\varepsilon Q,\ k(\theta)=\varepsilon K(\theta),\ \varepsilon\ll 1$$
$$M_i=\varepsilon^2 M_i^*(P, Q, r, \psi, \theta, \varphi, t)\ \ (i=1, 2),\ M_3=\varepsilon M_3^*(P, Q, r, \psi, \theta, \varphi, t) \tag{1.3}$$

Paper [1] also considered perturbed motion of a heavy rigid body, close to the Lagrange case. The conditions for possible averaging of the equations of motion with respect to the angle of nutation were given, and an averaged system of equations was obtained. A numerical solution of the averaged system was developed for the case of linear dissipative moments. In contrast to [1], studies [2,3] considered the case of a body that rotates rapidly about the axis of dynamic symmetry*, and therefore the generating solution was not the trajectory of motion in the Lagrange case, but rather some simpler solution. As a consequence, explicit analytic solutions could be obtained via the averaging method in first and second approximations.

In [2], as in this study, conditions (1.2) and (1.3) were assumed to be observed. In contrast to this study, it was assumed in [2] that the body is acted upon by a restoring moment whose maximum value is equal to k, and which is generated by a force of constant magnitude and direction, applied at some fixed point of the axis of dynamic symmetry.

In contrast to the third inequality in (1.2), it was assumed in [3] that the perturbing moments are small as compared to the restoring moment, $|M_i| \ll k$ $(i = 1, 2, 3)$*.

Perturbed rotational motion of a rigid body, close to pseudo-regular precession in the Lagrange case, was investigated earlier**.

The new variables P and Q, as well as the variables and constants r, ψ, θ, ϕ, K, A, C, M_i^* $(i = 1, 2, 3)$, are assumed to be bounded quantities of order unity as $\epsilon \to 0$.

The problem that we pose is that of investigating the asymptotic behavior of the solutions of system (1.1) for small ϵ, if conditions (1.2) and (1.3) are satisfied. We will employ the averaging method [4-6] on a time interval of order ϵ^{-1}.

The averaging method is extensively employed in problems of rigid-body dynamics. The method was used in [6-8] to investigate some problems of dynamics, in particular for dynamically symmetrical bodies. Averaging with respect to Euler-Poinsot motion was first performed in [9] for an asymmetrical body. Perturbed motion of a rigid body, close to the Lagrange case, was investigated in [1-3,6,8,10]. Simplifying assumptions (1.2) or (1.3) make it possible to obtain a fairly simple averaging scheme in the general case, and to investigate a number of examples.

2. AVERAGING PROCEDURE

We make the change of variables (1.3) in system (1.1). Cancelling by ϵ on both sides of the first two equations in (1.1), we obtain

$$AP^{\cdot}+(C-A)Qr=K(\theta)\sin\theta\cos\varphi+\varepsilon M_1^*$$
$$AQ^{\cdot}+(A-C)Pr=-K(\theta)\sin\theta\sin\varphi+\varepsilon M_2^*,\ Cr^{\cdot}=\varepsilon M_3^*$$
$$\psi^{\cdot}=\varepsilon(P\sin\varphi+Q\cos\varphi)\operatorname{cosec}\theta,\ \theta^{\cdot}=\varepsilon(P\cos\varphi-Q\sin\varphi) \tag{2.1}$$
$$\varphi^{\cdot}=r-\varepsilon(P\sin\varphi+Q\cos\varphi)\operatorname{ctg}\theta$$

First we consider the zero-approximation solution, and we set $\epsilon = 0$ in (2.1). The last four equations in (2.1) yield

$$r=r_0,\ \psi=\psi_0,\ \theta=\theta_0,\ \varphi=r_0t+\varphi_0 \tag{2.2}$$

*D. D. Leshchenko and S. N. Sallam, "Perturbed rotational motion of a rigid body with mass distribution close to the Lagrange case," Odessa, 1988; UkrNIINTI Deposition No. 1655-Uk88, 28 June 1988.

**D. D. Leshchenko and S. N. Sallam, "Perturbed motion, close to pseudo-regular precession, of a rigid body," Odessa, 1988; UkrNIINTI Deposition No. 1656-Uk88, 28 June 1988.

Here r_0, ψ_0, θ_0, ϕ_0 are constants equal to the initial values of the variables for t = 0. We substitute (2.2) into the first two equations in (2.1) for $\epsilon = 0$, and we integrate the resultant system of linear equations for P and Q. We write the solution as follows:

$$
\begin{gathered}
P=a\cos\gamma_0+b\sin\gamma_0+K_0C^{-1}r_0^{-1}\sin\theta_0\sin(r_0t+\varphi_0)\\
Q=a\sin\gamma_0-b\cos\gamma_0+K_0C^{-1}r_0^{-1}\sin\theta_0\cos(r_0t+\varphi_0)\\
a=P_0-K_0C^{-1}r_0^{-1}\sin\theta_0\sin\varphi_0,\ b=-Q_0+K_0C^{-1}\sin\theta_0\cos\varphi_0r_0^{-1}\\
\gamma_0=n_0t,\ n_0=(C-A)A^{-1}r_0\neq0,\ |n_0/r_0|\leqslant1,\ K_0=K(\theta_0)
\end{gathered}
\tag{2.3}
$$

Here P_0 and Q_0 are the initial values of the new variables P and Q, introduced in accordance with (1.3), while the variable $\gamma = \gamma_0$ is understood as the phase of the vibrations. System (2.1) is essentially nonlinear (the frequency of natural vibration of the variables P and Q depends on the slow variable r), and therefore in what follows we introduce the additional variable γ, defined by the equation

$$\gamma^{\cdot}=n,\ \gamma(0)=0,\ n=(C-A)A^{-1}r \tag{2.4}$$

For $\epsilon = 0$ we have $\gamma = \gamma_0 = n_0t$ in accordance with (2.3). Equations (2.2) and (2.3) define the general solution of system (2.1), (2.4) for $\epsilon = 0$. By eliminating constants and allowing for (2.2), the first two expressions in (2.3) can be rewritten in the following equivalent form:

$$
\begin{gathered}
P=a\cos\gamma+b\sin\gamma+KC^{-1}r^{-1}\sin\theta\sin\varphi\\
Q=a\sin\gamma-b\cos\gamma+KC^{-1}r^{-1}\sin\theta\cos\varphi
\end{gathered}
\tag{2.5}
$$

and can be solved for a and b:

$$
\begin{gathered}
a=P\cos\gamma+Q\sin\gamma-KC^{-1}r^{-1}\sin\theta\sin(\gamma+\varphi)\\
b=P\sin\gamma-Q\cos\gamma+KC^{-1}r^{-1}\sin\theta\cos(\gamma+\varphi)
\end{gathered}
\tag{2.6}
$$

We consider system (2.1) for $\epsilon \neq 0$, and expressions (2.5) and (2.6) as change-of-variable formulas (containing the variable γ), which specify the conversion from variables P and Q to the Van der Pol variables a and b and vice versa [6]. Using these formulas, we convert in system (2.1), (2.4) from the variables P, Q, r, ψ, θ, ϕ, γ to the new variables a, b, r, ψ, θ, α, γ, where

$$\alpha=\gamma+\varphi \tag{2.7}$$

After performing the manipulations, we obtain a system of seven equations (instead of the six in (2.1)) that is more convenient for what follows:

$$
\begin{gathered}
a^{\cdot}=\varepsilon A^{-1}(M_1^0\cos\gamma+M_2^0\sin\gamma)-\varepsilon KC^{-1}r^{-1}\cos\theta(b-KC^{-1}r^{-1}\sin\theta\cos\alpha)+\\
+\varepsilon KC^{-2}r^{-2}M_3^0\sin\theta\sin\alpha-\varepsilon C^{-1}r^{-1}\sin\theta\sin\alpha(a\cos\alpha+b\sin\alpha)\,dK/d\theta\\
b^{\cdot}=\varepsilon A^{-1}(M_1^0\sin\gamma-M_2^0\cos\gamma)+\varepsilon KC^{-1}r^{-1}\cos\theta(a+KC^{-1}r^{-1}\sin\theta\sin\alpha)-\\
-\varepsilon KC^{-2}r^{-2}M_3^0\sin\theta\cos\alpha+\varepsilon C^{-1}r^{-1}\sin\theta\cos\alpha(a\cos\alpha+b\sin\alpha)\,dK/d\theta\\
r^{\cdot}=\varepsilon C^{-1}M_3^0,\ \psi^{\cdot}=\varepsilon\operatorname{cosec}\theta(a\sin\alpha-b\cos\alpha)+\varepsilon KC^{-1}r^{-1}\\
\alpha^{\cdot}=CA^{-1}r-\varepsilon\operatorname{ctg}\theta(a\sin\alpha-b\cos\alpha)-\varepsilon KC^{-1}r^{-1}\cos\theta\\
\theta^{\cdot}=\varepsilon(a\cos\alpha+b\sin\alpha),\ \gamma^{\cdot}=(C-A)A^{-1}r
\end{gathered}
\tag{2.8}
$$

Here, the M_i^0 denote functions obtained from M_i^* (see (1.3)) as a result of substitution (2.5)-(2.7), i.e.,

$$M_i^0(a,b,r,\psi,\theta,\alpha,\gamma,t)=M_i^*(P,Q,r,\psi,\theta,\varphi,t)\quad(i=1,2,3) \tag{2.9}$$

We should note that the conversion from the two variables P and Q to the three variables a, b, γ stems from considerations of convenience; for $\epsilon = 0$ the system for P and Q has the form of a linear system, while substitution (2.5) is nonsingular for all a and b.

We introduce vector x, whose components are provided by the slow variables a, b, r, ψ, θ of system (2.8). Then this system can be written as follows:

$$x^{\cdot}=\varepsilon X(x,\ \alpha,\ \gamma,\ t),\quad \alpha^{\cdot}=CA^{-1}r+\varepsilon Y(x,\ \alpha)$$
$$\gamma^{\cdot}=(C-A)A^{-1}r,\quad x(0)=x_0,\quad \alpha(0)=\alpha_0,\quad \gamma(0)=0 \tag{2.10}$$

Here vector-valued function X and scalar function Y are defined by the right sides of (2.8); the initial values are obtained in accordance with (2.2)-(2.4) and (2.7).

Let us consider system (2.8) or (2.10) from the standpoint of employing the averaging method [4-6]. System (2.8) contains the slow variables a, b, r, ψ, θ, and also fast variables, namely the phases α and γ and the time t, with γ appearing only in the first three equations of (2.8). The system is essentially nonlinear, and direct use of the averaging method is extremely difficult [11]. For simplicity, we will assume that the perturbing moments M_i^* are independent of t. Since the M_i^* (i = 1, 2, 3) are 2π-periodic in ϕ, in accordance with substitution (2.5)-(2.7) we have that functions M_i^0 from (2.9) will be 2π-periodic functions of α and γ. Then system (2.10) contains two rotating phases α and γ, and the corresponding frequencies $CA^{-1}r$ and $(C - A)A^{-1}r$ are variable. In averaging system (2.8) or (2.10), we should distinguish two cases: the nonresonant case, when frequencies $CA^{-1}r$ and $(C - A)A^{-1}r$ are not commensurable; and the resonant case, when they are commensurable [11]. A very important feature of system (2.10) is the fact that the frequency ratio is constant: $[(C-A)A^{-1}r]/[CA^{-1}r]=1-AC^{-1}$, and the case of resonance occurs for

$$C/A=i/j,\quad i/j\leqslant 2 \tag{2.11}$$

where i and j are natural relatively prime numbers; in the nonresonant case C/A is an irrational number. As a result of (2.11), averaging of nonlinear system (2.10), in which X is independent of t, is equivalent to averaging of a quasi-linear system with constant frequencies. This can be achieved by introducing the independent variable γ.

In the nonresonant case ($C/A \neq i/j$), we obtain the first-approximation averaged system by independent averaging of the right sides of system (2.8) with respect to both fast variables α, γ. As a result, we obtain the following equations for the slow variables:

$$a^{\cdot}=\varepsilon A^{-1}\mu_1-\varepsilon bKC^{-1}r^{-1}\cos\theta+\varepsilon KC^{-2}r^{-2}\sin\theta\mu_3^{s}-{}^1/_2\varepsilon C^{-1}r^{-1}b\sin\theta dK/d\theta$$
$$b^{\cdot}=\varepsilon A^{-1}\mu_2+\varepsilon aKC^{-1}r^{-1}\cos\theta-\varepsilon KC^{-2}r^{-2}\sin\theta\mu_3^{c}+{}^1/_2\varepsilon C^{-1}r^{-1}a\sin\theta dK/d\theta$$
$$r^{\cdot}=\varepsilon C^{-1}\mu_3,\quad \psi^{\cdot}=\varepsilon KC^{-1}r^{-1},\quad \theta^{\cdot}=0$$

$$\mu_1(a,b,r,\psi,\theta)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}(M_1^0\cos\gamma+M_2^0\sin\gamma)\,d\alpha\,d\gamma$$

$$\mu_2(a,b,r,\psi,\theta)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}(M_1^0\sin\gamma-M_2^0\cos\gamma)\,d\alpha\,d\gamma,$$

$$\mu_3(a,b,r,\psi,\theta)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}M_3^0\,d\alpha\,d\gamma \tag{2.12}$$

$$\mu_3^{s}(a,b,r,\psi,\theta)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}M_3^0\sin\alpha\,d\alpha\,d\gamma,\qquad \mu_3^{c}(a,b,r,\psi,\theta)=$$

$$=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}M_3^0\cos\alpha\,d\alpha\,d\gamma$$

Solving averaged system (2.12) for perturbing moments of specific form, we can determine the motion of the body in the nonresonant case with an error of order ϵ on a time interval of order ϵ^{-1}. We should note that the last equation in (2.12) can be integrated; it yields $\theta = \theta_0$.

The above system is equivalent to a two-frequency system with constant frequencies, since both frequencies are proportional to the axial component r of the angular velocity vector. Therefore the applicability of the averaging method can be substantiated in the same way as for a quasi-linear system [2].

In the resonant case (2.11), system (2.10) is a single-frequency system. We replace α by a new slow variable that comprises a linear combination of the phase with integer coefficients:

$$\lambda=\alpha-i(i-j)^{-1}\gamma,\ i/j\neq 1,\ i/j\leq 2,\ i,j>0 \tag{2.13}$$

System (2.10) assumes the form of a standard system with rotating phase:

$$\begin{gathered} x^{\cdot}=\varepsilon X(x,\ i(i-j)^{-1}\gamma+\lambda,\ \gamma) \\ \lambda^{\cdot}=\varepsilon Y(x,\ i(i-j)^{-1}\gamma+\lambda),\ \gamma^{\cdot}=(C-A)A^{-1}r \end{gathered} \tag{2.14}$$

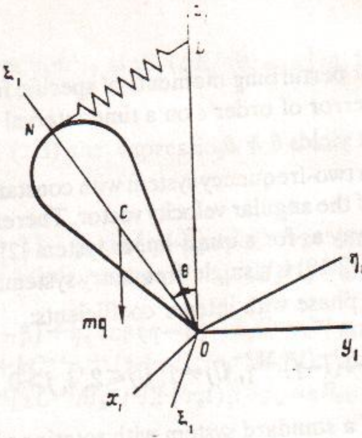
where its right sides are $2|i-j|\pi$-periodic in γ. We can set up the first-approximation system by averaging the right sides of (2.14) with respect to this period of variation of the argument γ. As a result, we obtain the following system of equations for the slow variables:

$$\begin{gathered} a^{\cdot}=\varepsilon A^{-1}\mu_1{}^*-\varepsilon KC^{-1}r^{-1}b\cos\theta+\varepsilon KC^{-2}r^{-2}\sin\theta\mu_3{}^{*s}-{}^1/_2\varepsilon C^{-1}r^{-1}b\sin\theta dK/d\theta \\ b^{\cdot}=\varepsilon A^{-1}\mu_2{}^*+\varepsilon KC^{-1}r^{-1}a\cos\theta-\varepsilon KC^{-2}r^{-2}\sin\theta\mu_3{}^{*c}+{}^1/_2\varepsilon C^{-1}r^{-1}a\sin\theta dK/d\theta \\ r^{\cdot}=\varepsilon C^{-1}\mu_3{}^* \\ \psi^{\cdot}=\varepsilon KC^{-1}r^{-1},\ \theta^{\cdot}=0,\ \lambda^{\cdot}=-\varepsilon KC^{-1}r^{-1}\cos\theta \\ \mu_1{}^*(a,b,r,\psi,\theta,\lambda)=\frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|}(M_1{}^0\cos\gamma+M_2{}^0\sin\gamma)\,d\gamma \\ \mu_2{}^*(a,b,r,\psi,\theta,\lambda)=\frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|}(M_1{}^0\sin\gamma-M_2{}^0\cos\gamma)\,d\gamma \\ \mu_3{}^*(a,b,r,\psi,\theta,\lambda)=\frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|}M_3{}^0\,d\gamma \\ \mu_3{}^{*s}(a,b,r,\psi,\theta,\lambda)=\frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|}M_3{}^0\sin\alpha\,d\gamma,\quad \mu_3{}^{*c}(a,b,r,\psi,\theta,\lambda)= \\ =\frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|}M_3{}^0\cos\alpha\,d\gamma \end{gathered} \tag{2.15}$$

It is assumed that, in the integrands, the variable α is replaced by λ in accordance with (2.13). We should note that the next-to-last equation in (2.15) has the solution $\theta = \theta_0$.

Solving averaged system (2.15) for perturbing moments of specific form, we can determine the motion of the body in the resonant case with an error of order ϵ on a time interval of order ϵ^{-1}. The substantiation procedure is the standard one [4,5].

As an example of a restoring moment that depends on the angle of nutation, let us consider a rigid body with a spring attached to it at point N, the end L of the spring being fixed. The body (see the accompanying figure) is acted upon by the force of gravity mg and the elastic force F of the spring, whose modulus is proportional to the deformation of the spring $F = \lambda_1(s - s_0)$. Here λ_1 is the stiffness of the spring. In this case the restoring moment has the form



$$k(\theta)=mgl+\lambda_1 hz[1-s_0(h^2+z^2-2hz\cos\theta)^{-1/2}] \qquad (2.16)$$

$$ON=z,\ \ OC=l,\ \ OL=h,\ \ LN=s=s(\theta)\ \ k(\theta)=\varepsilon K(\theta)$$

In what follows, using the technique described above, we consider some specific examples of perturbed motion of a rigid body.

3. CASE OF SMALL CONSTANT MOMENT

Consider motion of a rigid body in the Lagrange case, in response to a small moment that is constant in the associated axes and is applied along the axis of symmetry. In this case the perturbing moments M_i (i = 1, 2, 3) have the form

$$M_1=M_2=0,\ M_3=\varepsilon M_3^*=\mathrm{const} \qquad (3.1)$$

Converting to new slow variables a, b, r, ψ, θ, in the nonresonant case we obtain an averaged system of type (2.12):

$$\begin{aligned} a^{\cdot}&=-\varepsilon KC^{-1}r^{-1}b\cos\theta-\\ &-{}^1/_2\varepsilon C^{-1}r^{-1}b\sin\theta dK/d\theta\\ b^{\cdot}&=\varepsilon KC^{-1}r^{-1}a\cos\theta+\\ &+{}^1/_2\varepsilon C^{-1}r^{-1}a\sin\theta dK/d\theta\\ r^{\cdot}&=\varepsilon C^{-1}M_3^*,\ \ \psi^{\cdot}=\varepsilon KC^{-1}r^{-1},\ \ \theta^{\cdot}=0 \end{aligned} \qquad (3.2)$$

Integrating the third equation in (3.2), we obtain

$$r=r_0+\varepsilon C^{-1}M_3^*t \qquad (3.3)$$

We substitute (3.3) into (3.2) and integrate the equation for $\psi^{\cdot}$:

$$\psi=\psi_0+K(M_3^*)^{-1}\ln|1+\varepsilon C^{-1}M_3^*r_0^{-1}t| \qquad (3.4)$$

Here ψ_0 and r_0 are arbitrary initial values of the angle of precession and of the axial velocity of rotation. As follows from (3.2), the angle of nutation θ does not vary over the time of motion of the body $\theta=\theta_0$.

After replacement of r by expression (3.3), the solution of the system comprising the first two equations in (3.2) can be written as follows:

$$\begin{aligned} a&=P_0\cos\beta+Q_0\sin\beta-K_0C^{-1}r_0^{-1}\sin\theta_0\sin(\beta+\varphi_0)\\ b&=P_0\sin\beta-Q_0\cos\beta+K_0C^{-1}r_0^{-1}\sin\theta_0\cos(\beta+\varphi_0) \end{aligned} \qquad (3.5)$$

$$\beta=(M_3^*)^{-1}[K_0\cos\theta_0+{}^1/_2\sin\theta_0(dK/d\theta)_{\theta=\theta_0}]\ln|1+\varepsilon C^{-1}r_0^{-1}M_3^*t|,\ K_0=K(\theta_0) \quad (3.5)$$

Substituting into (2.5) and (1.3) the expressions for a and b from (3.5), and for r from (3.3), we can determine the following:

$$\begin{gathered} p=p_0\cos(\gamma-\beta)-q_0\sin(\gamma-\beta)+k_0C^{-1}r_0^{-1}\sin\theta_0\sin(\gamma-\beta-\varphi_0)+ \\ +kC^{-1}r_0^{-1}(1+\varepsilon C^{-1}r_0^{-1}M_3^*t)^{-1}\sin\theta_0\sin\varphi \\ q=p_0\sin(\gamma-\beta)+q_0\cos(\gamma-\beta)-k_0C^{-1}r_0^{-1}\sin\theta_0\cos(\gamma-\beta-\varphi_0)+ \\ +kC^{-1}r_0^{-1}(1+\varepsilon C^{-1}r_0^{-1}M_3^*t)^{-1}\sin\theta_0\cos\varphi \\ \gamma=(C-A)A^{-1}(\varepsilon C^{-1}M_3^*t^2/2+r_0t),\quad p_0=\varepsilon P_0,\quad q_0=\varepsilon Q_0,\quad k_0=\varepsilon K(\theta_0) \end{gathered} \quad (3.6)$$

For the example under consideration of a body with a spring attached to it, the expressions for p and q can be obtained from (3.6), with k replaced by 2.16), and k_0 replaced by the same expression for $\theta = \theta_0$. Here

$$\begin{gathered} \beta=(\varepsilon M_3^*)^{-1}[k_0\cos\theta_0+\lambda_1h^2z^2s_0\sin^2\theta_0(h^2+z^2-2hz\cos\theta_0)^{-3/2}]\ln|1+ \\ +\varepsilon C^{-1}r_0^{-1}M_3^*t|. \end{gathered}$$

In accordance with (3.3), the quantity $|r(\tau)|$, $\theta = \epsilon t$, increases if the parameters r_0 and M_3^* are of the same sign; and decreases if the signs are different. The angle of precession ψ in (3.4) contains a variable component whose modulus increases monotonically in both cases; in the first case it is bounded for finite $\tau \sim 1$, while in the second it tends to infinity as $\tau \to -Cr_0/M_3^*$; here $r \to 0$.

The variable β in (3.5) and (3.6) varies similarly to ψ if $\theta_0 \neq \pm\pi/2$. The slow variables a and b are bounded 2π-periodic functions of β.

In accordance with (3.6), the components p and q of the angular velocity vector contain bounded oscillating terms that result from the nonzero initial data p_0 and q_0; and also terms that result from the restoring moment (1.2) and (2.16).

We should note that, when the spring is not present, comparison of the resultant expressions (3.5) and (3.6) for the slow variables a, b, p, q with the corresponding formulas of [2], if we formally set $\lambda_1 = 0$ in them, yields coinciding expressions.

It should be emphasized that formulas (3.6) for p and q contain variable and constant components of the restoring moments $k(\theta)$ and $k_0 = k(\theta_0)$.

4. CASE OF LINEAR EXTERNAL DISSIPATIVE MOMENTS

Consider perturbed Lagrange motion with allowance for the moments acting on the rigid body from the external environment. We will assume that the perturbing moments M_i (i = 1, 2, 3) are linear-dissipative [12]:

$$M_1=-\varepsilon I_1p,\ M_2=-\varepsilon I_1q,\ M_3=-\varepsilon I_3r,\ I_1,\ I_3>0 \quad (4.1)$$

Here I_1 and I_3 are constant proportionality factors that depend on the properties of the medium and the shape of the body.

We write the perturbing moments with allowance for expressions (1.3) for p and q:

$$M_1=-\varepsilon^2I_1P,\ M_2=-\varepsilon^2I_1Q,\ M_3=-\varepsilon I_3r \quad (4.2)$$

In accordance with § 2, we convert to new slow variables a, b, r, ψ, θ, and we obtain averaged system (2.12) of the form

$$\begin{gathered} a^{\cdot}=-\varepsilon I_1A^{-1}a-\varepsilon C^{-1}r^{-1}b(K\cos\theta+{}^1/_2\sin\theta dK/d\theta) \\ b^{\cdot}=-\varepsilon I_1A^{-1}b+\varepsilon C^{-1}r^{-1}a(K\cos\theta+{}^1/_2\sin\theta dK/d\theta) \\ r^{\cdot}=-\varepsilon I_3C^{-1}r,\ \psi^{\cdot}=\varepsilon KC^{-1}r^{-1},\ \theta^{\cdot}=0 \end{gathered} \quad (4.3)$$

Integrating the third equation in (4.3), we obtain (r_0 is the arbitrary initial value of the axial rotational velocity)

$$r = r_0 \exp(-\varepsilon I_3 C^{-1} t), \quad r_0 \neq 0 \tag{4.4}$$

Equation (4.3) for $\psi^{\cdot}$ can be integrated, with allowance for (4.4); it yields (ψ_0 is a constant equal to the initial value of the precession angle for t = 0):

$$\psi = \psi_0 + K I_3^{-1} r_0^{-1} [\exp(\varepsilon I_3 C^{-1} t) - 1] \tag{4.5}$$

In addition, as can be seen from (4.3), the angle of nutation maintains constant value $\theta = \theta_0$. Replacing r by (4.4) in the first two equations of (4.3), we obtain a system of the form

$$\begin{aligned} a^{\cdot} &= -\varepsilon I_1 A^{-1} a - \varepsilon C^{-1} r_0^{-1} b \exp(\varepsilon I_3 C^{-1} t)(K \cos\theta + {}^1/_2 \sin\theta \, dK/d\theta) \\ b^{\cdot} &= -\varepsilon I_1 A^{-1} b + \varepsilon C^{-1} r_0^{-1} a \exp(\varepsilon I_3 C^{-1} t)(K \cos\theta + {}^1/_2 \sin\theta \, dK/d\theta) \end{aligned}$$

whose solution can be written as follows in accordance with [13] (p. 534):

$$\begin{aligned} a &= \exp(-\varepsilon I_1 A^{-1} t)[P_0 \cos\eta + Q_0 \sin\eta - K_0 C^{-1} r_0^{-1} \sin\theta_0 \sin(\eta + \varphi_0)] \\ b &= \exp(-\varepsilon I_1 A^{-1} t)[P_0 \sin\eta - Q_0 \cos\eta + K_0 C^{-1} r_0^{-1} \sin\theta_0 \cos(\eta + \varphi_0)] \\ \eta &= r_0^{-1} I_3^{-1} (K \cos\theta + {}^1/_2 \sin\theta \, dK/d\theta)[\exp(\varepsilon I_3 C^{-1} t) - 1], \quad K_0 = K(\theta_0) \end{aligned} \tag{4.6}$$

As a result of substitution of the expressions for a and b from (4.6), and for r from (4.4), into expressions (2.5) and (1.3) for P, Q, p, q, we have

$$\begin{aligned} p &= \exp(-\varepsilon I_1 A^{-1} t)[p_0 \cos(\gamma - \eta) - q_0 \sin(\gamma - \eta) + k_0 C^{-1} r_0^{-1} \sin\theta_0 \sin(\gamma - \\ &\quad - \eta - \varphi_0)] + k C^{-1} r_0^{-1} \exp(\varepsilon I_3 C^{-1} t) \sin\theta_0 \sin\varphi \\ q &= \exp(-\varepsilon I_1 A^{-1} t)[p_0 \sin(\gamma - \eta) + q_0 \cos(\gamma - \eta) - k_0 C^{-1} r_0^{-1} \sin\theta_0 \cos(\gamma - \\ &\quad - \eta - \varphi_0)] + k C^{-1} r_0^{-1} \exp(\varepsilon I_3 C^{-1} t) \sin\theta_0 \cos\varphi \\ \gamma &= \frac{C}{I_3} \frac{C - A}{A} \frac{r_0}{\varepsilon} [1 - \exp(-\varepsilon I_3 C^{-1} t)], \quad p_0 = \varepsilon P_0 \\ & q_0 = \varepsilon Q_0, \quad k_0 = \varepsilon K_0 \end{aligned} \tag{4.7}$$

We should note that, in contrast to the corresponding formulas of [2], expressions (4.7) contain a constant component k_0 and a variable component of the perturbing moment $k\theta$).

For a body with a spring, we need to substitute into (4.7) expression (2.16) for the restoring moment $k = k(\theta)$, and this same expression for $\theta = \theta_0$ for $k_0 = k(\theta_0)$. In this case

$$\begin{aligned} \eta = r_0^{-1} I_3^{-1} \varepsilon^{-1} [k \cos\theta + {}^1/_2 \lambda_1 h^2 z^2 \sin^2\theta (h^2 + z^2 - \\ - 2hz \cos\theta)^{-3/2}][\exp(\varepsilon I_3 C^{-1} t) - 1] \end{aligned}$$

Thus, we have developed a solution of the first-approximation system for the slow variables in the case of dissipative moment (4.1). We should note certain qualitative features of motion in this case. The modulus of the axial rotational velocity r decreases monotonically in exponential fashion in accordance with (4.4). The increment of the precession angle $\psi - \psi_0$ increases exponentially slowly in accordance with (4.5). It follows from (4.6) that the slow variables a and b tend monotonically to zero in exponential fashion.

In accordance with (4.7), the terms of the projections p and q, resulting from the initial values p_0 and q_0, attenuate exponentially. At the same time, the projections p and q contain exponentially increasing terms that are proportional to the restoring moment, thus leading to an exponential increase in $(p^2 + q^2)^{1/2}$.

Comparison of the resultant expressions (4.6) and (4.7) for the slow variables a, b, p, q with the corresponding formulas of [2], the spring being absent (we formally set $\lambda_1 = 0$), indicates that the expressions coincide.

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