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ROTATIONS OF A RIGID BODY UNDER THE ACTION OF UNSTEADY RESTORING AND PERTURBATION TORQUES

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Perturbed rotations of a rigid body close to Lagrange's regular precession are investigated in the case where the body is acted upon by a restoring and perturbation torques slowly varying in time. The body is assumed to be spun up to a high angular velocity and the restoring and perturbation torques are assumed to be small, with a certain hierarchy of smallness of the components. An averaged first-approximation system of equations of motion is obtained for the essentially nonlinear system under consideration in non-resonant and resonant cases. Examples of motion of the body are considered for specific cases of restoring, perturbation, and control torques.

1. STATEMENT OF THE PROBLEM

Consider the motion of a dynamically symmetric rigid body about a fixed point O under the action of a restoring and perturbation torques depending on the slow time $\tau = \varepsilon t$, where t is time and ε is a small parameter ($\varepsilon \ll 1$). The equations of motion have the form

$$\begin{gathered} A\dot{p} + (C-A)qr = k(\tau)\sin\theta\cos\varphi + M_1, \quad A\dot{q} + (A-C)pr = -k(\tau)\sin\theta\sin\varphi + M_2, \quad C\dot{r} = M_3, \\ \dot{\psi} = (p\sin\varphi + q\cos\varphi)\operatorname{cosec}\theta, \quad \dot{\theta} = p\cos\varphi - q\sin\varphi, \quad \dot{\varphi} = r - (p\sin\varphi + q\cos\varphi)\cot\theta, \\ M_l = M_l(p, q, r, \psi, \theta, \varphi, \tau), \quad \tau = \varepsilon t \qquad (l = 1, 2, 3), \end{gathered} \tag{1.1}$$

where p, q, and r are the projections of the angular velocity of the body onto the principal axes of inertia passing through the point O, M_l are the projections of the perturbation torques onto these axes, and A and C are, respectively, the axial and equatorial moments of inertia of the body at the point O ($A \neq C$). The torques M_l depend on the slow time τ and are 2π-periodic functions of the Eulerian angles ψ, φ, and θ. We assume that the body is acted upon also by a restoring torque depending on the slow time, $k(\tau)$. If the perturbation torque is zero, $M_l \equiv 0$, and the restoring torque is time-independent, $k(\tau) = \text{const}$, the equations of (1.1) correspond to Lagrange's case.

We assume the following conditions to hold:

$$\sqrt{p^2 + q^2} \ll r, \quad Cr^2 \gg k, \quad |M_{1,2}| \ll k, \quad M_3 \sim k. \tag{1.2}$$

The relations of (1.2) imply that (i) the direction of the angular velocity is close to that of the symmetry axis of the body, (ii) the projection of the angular velocity of the body onto the symmetry axis is large enough, so that the kinetic energy of the body is much greater than the potential energy due to the restoring torque, and (iii) two projections of the perturbation torque onto the principal axes of inertia of the body are small as compared with the restoring torque, while the third projection coincides with the restoring torque in order of magnitude.

The inequalities of (1.2) allow one to introduce the following relations:

$$\begin{gathered} p = \varepsilon P, \quad q = \varepsilon Q, \quad k(\tau) = \varepsilon K(\tau), \quad \tau = \varepsilon t, \\ M_{1,2} = \varepsilon^2 M_{1,2}^*(P, Q, r, \psi, \theta, \varphi, \tau), \quad M_3 = \varepsilon M_3^*(P, Q, r, \psi, \theta, \varphi, \tau). \end{gathered} \tag{1.3}$$

Previously, fast rotations of a rigid body close to Lagrange's case have been considered for the restoring torque constant in magnitude, $k = \text{const}$ [1], and for the restoring torque depending on the nutation angle, $k = k(\theta)$ [2]. The influence of the perturbation torque accounted for by the resistance of the medium and the perturbation torque constant in the body-attached reference frame have been analyzed.

In the present paper, we investigate the motion of a rigid body under the action of the perturbation and restoring torques depending on the slow time τ, i.e., $k = k(\tau)$, $M_l = M_l(p, q, r, \psi, \theta, \varphi, \tau)$. The new variables P and Q, as well as the variables r, ψ, θ, and φ, the functions K and M_l^* (l = 1,2,3), and the parameters A and C are assumed to be bounded quantities having an order of unity as $\varepsilon \to 0$.

We will investigate the asymptotic behavior of system (1.1) with small ε on the time interval of the order of ε^{-1}, provided that the conditions of (1.2) and (1.3) are satisfied. For this investigation, we will utilize the method of averaging [3, 4]. The method of averaging has been widely utilized when solving problems of dynamics of a rigid body. The simplifying assumptions of (1.2) or (1.3) enable one to obtain a fairly simple averaging scheme in the general case and solve a number of example problems.

2. CONSTRUCTION OF THE AVERAGED EQUATIONS OF MOTION

We subject system (1.2) to the change of variables in accordance with (1.3). Then we cancel ε in both sides of the first two equations of (1.1) to obtain

$$A\dot{P} + (C-A)Qr = K(\tau)\sin\theta\cos\varphi + \varepsilon M^*, \quad A\dot{Q} + (A-C)Pr = -K(\tau)\sin\theta\sin\varphi + \varepsilon M_2^*, \quad C\dot{r} = \varepsilon M_3^*,$$
$$\dot{\psi} = \varepsilon(P\sin\varphi + Q\cos\varphi)\operatorname{cosec}\theta, \quad \dot{\varphi} = r - \varepsilon(P\sin\varphi + Q\cos\varphi)\cot\theta, \quad \dot{\theta} = \varepsilon(P\cos\varphi - Q\sin\varphi). \tag{2.1}$$

According to the terminology of [3, 4], system (2.1) is a two-frequency essentially nonlinear system.

Consider first the first-approximation system obtained by setting $\varepsilon = 0$ in (2.1). From the last four equations we find

$$r = r_0, \quad \psi = \psi_0, \quad \theta = \theta_0, \quad \varphi = r_0 t + \varphi_0, \quad K = K(\tau), \quad \tau = \text{const}, \tag{2.2}$$

where $r_0, \varphi_0, \theta_0, \varphi_0$ are constants equal to the values of the respective variables at $t = 0$. Substitute the relations of (2.2) into the first two equations of (2.1) with $\varepsilon = 0$ and then integrate the resulting system of linear equations for P and Q to obtain

$$P = a\cos\gamma + b\sin\gamma + \frac{K}{Cr}\sin\theta\sin\varphi, \quad Q = a\sin\gamma - b\cos\gamma + \frac{K}{Cr}\sin\theta\cos\varphi, \tag{2.3}$$
$$a = P - \frac{K}{Cr}\sin\theta\sin\alpha, \quad b = -Q + \frac{K}{Cr}\sin\theta\cos\alpha,$$
$$\dot{\gamma} = n, \quad \gamma(0) = 0, \quad n = \frac{C-A}{A}r \neq 0, \quad \left|\frac{n}{r}\right| \leq 1, \quad \alpha = \varphi + \gamma,$$

where a and b are osculating (Van der Pol type) variables introduced instead of those of (1.3) and γ plays the role of the phase of oscillations.

We will utilize system (2.1) with $\varepsilon = 0$ and the relations of (2.3) as relations (containing the variable γ) that define a single-valued transformation between the variables (P, Q) and (a, b). Using these relations, we proceed from the variables $(P, Q, r, \psi, \theta, \varphi, \tau)$ to the variables $(a, b, r, \psi, \theta, \alpha, \gamma, \tau)$. Note that the phases φ, α, and γ are connected by a finite relation which turns out to be more convenient for the further analysis of the standard system with two rotating phases γ and α. After appropriate transformation we obtain the system

$$\dot{a} = \frac{\varepsilon}{A}(M_1^0\cos\gamma + M_2^0\sin\gamma) + \frac{\varepsilon K(\tau)}{C^2r^2}M_3^0\sin\theta\sin\alpha - \frac{\varepsilon K(\tau)}{Cr}\cos\theta\left[b - \frac{K(\tau)}{Cr}\sin\theta\cos\alpha\right] - \frac{\varepsilon K'(\tau)}{Cr}\sin\theta\cos\alpha,$$
$$\dot{b} = \frac{\varepsilon}{A}(M_1^0\sin\gamma - M_2^0\cos\gamma) - \frac{\varepsilon K(\tau)}{C^2r^2}M_3^0\sin\theta\cos\alpha + \frac{\varepsilon K(\tau)}{Cr}\cos\theta\left[a + \frac{K(\tau)}{Cr}\sin\theta\sin\alpha\right] + \frac{\varepsilon K'(\tau)}{Cr}\sin\theta\sin\alpha, \tag{2.4}$$
$$\dot{r} = \frac{\varepsilon}{C}M_3^0, \quad \dot{\theta} = \varepsilon(a\cos\alpha + b\sin\alpha), \quad \dot{\psi} = \varepsilon(a\sin\alpha - b\cos\alpha)\operatorname{cosec}\theta + \frac{\varepsilon K(\tau)}{Cr},$$
$$\dot{\alpha} = \frac{C}{A}r - \varepsilon(a\sin\alpha - b\cos\alpha)\cot\theta - \frac{\varepsilon K(\tau)}{Cr}\cos\theta, \quad \dot{\gamma} = \frac{C-A}{A}r,$$

$$M_l^0(a,b,r,\psi,\theta,\alpha,\gamma,\tau) = M_l^*(P,Q,r,\psi,\theta,\varphi,\tau) \qquad (l=1,2,3),$$

which is more convenient for the subsequent investigation.

Note that if K = const and M_l do not depend on ε, system (2.4) coincides with the respective system of [1].

Let us investigate the possibility of the method of averaging to be applied to system (2.4). This system contains slow variables, a, b, r, ψ, θ, and τ, and fast variables (phases), α and γ. The restoring torque in system (2.4) depends on the slow variable τ, which leads to the appearance of terms containing the derivative $\varepsilon K'(\tau)$ in the first two equations. If the perturbation torques depend on time t, then the system is essentially nonlinear and the application of the averaging method is rather complicated. We will consider a simpler case where the perturbation torques depend on the slow time $\tau = \varepsilon t$.

The torques M_l^* are 2π-periodic in φ and, therefore, the M_l^0 are 2π-periodic functions of α and γ. In this case, system (2.4) has two rotating phases, α and γ, and the frequencies $\omega_\alpha = CA^{-1}r$ and $\omega_\gamma = (C-A)A^{-1}r$ corresponding to these phases depend on the slow variable. When averaging system (2.4), one should distinguish between the non-resonant case, where the frequencies ω_α and ω_γ are incommensurable (C/A is an irrational number), and the resonant case, where these frequencies are commensurable ($C/A = i/j$, $i/j \le 2$, i and j are coprime positive integers). Since the frequency ratio is constant, $\omega_\gamma/\omega_\alpha = 1 - AC^{-1}$, introducing the variable γ makes the averaging of system (2.4) equivalent to the averaging of a quasi-linear system with constant frequencies.

In the non-resonant case ($C/A \ne i/j$), one can obtain the first-approximation averaged system by averaging the right-hand sides of (2.4) independently with respect to both of the fast variables α and γ. Then we perform the change of variables $\tau = \varepsilon t$ ($\varepsilon \ll 1$; t is time) and cancel ε in both sides of the resulting equations to obtain

$$\begin{aligned}
&a' = \frac{\mu_1}{A} - \frac{bK(\tau)}{Cr}\cos\theta + \frac{K(\tau)}{C^2r^2}\sin\theta\,\mu_3^{\mathrm{s}}, \quad b' = \frac{\mu_2}{A} + \frac{aK(\tau)}{Cr}\cos\theta - \frac{K(\tau)}{C^2r^2}\sin\theta\,\mu_3^{\mathrm{c}},\\
&r' = \frac{\mu_3}{C}, \quad \psi' = \frac{K(\tau)}{Cr}, \quad \theta' = 0,\\
&\mu_1 = \frac{1}{4\pi^2}\int_0^{2\pi}\!\!\int_0^{2\pi}(M_1^0\cos\gamma + M_2^0\sin\gamma)\,d\alpha\,d\gamma, \quad \mu_2 = \frac{1}{4\pi^2}\int_0^{2\pi}\!\!\int_0^{2\pi}(M_1^0\sin\gamma - M_2^0\cos\gamma)\,d\alpha\,d\gamma,\\
&\mu_3 = \frac{1}{4\pi^2}\int_0^{2\pi}\!\!\int_0^{2\pi} M_3^0\,d\alpha\,d\gamma,\\
&\mu_3^{\mathrm{s}} = \frac{1}{4\pi^2}\int_0^{2\pi}\!\!\int_0^{2\pi} M_3^0\sin\alpha\,d\alpha\,d\gamma, \quad \mu_3^{\mathrm{c}} = \frac{1}{4\pi^2}\int_0^{2\pi}\!\!\int_0^{2\pi} M_3^0\cos\alpha\,d\alpha\,d\gamma, \qquad (\cdot)' = \frac{d(\cdot)}{d\tau}.
\end{aligned} \tag{2.5}$$

In the resonant case, system (2.4) is single-frequency. Instead of α, we introduce a slow variable λ defined as a linear combination of the phases with integer coefficients,

$$\lambda = \alpha - \frac{i}{i-j}\gamma, \qquad \frac{i}{j} \ne 1, \quad \frac{i}{j} \le 2, \quad i,j > 0. \tag{2.6}$$

Then the system takes a standard form with the rotating phase γ of which the right-hand sides of the resulting system are $2|i-j|\pi$-periodic functions. To construct the first-approximation system we average the right-hand sides of system (2.4) with respect to γ over the aforementioned period. The change of variables $\tau = \varepsilon t$ reduces the system to the form

$$\begin{aligned}
&a' = \frac{\mu_1^*}{A} - \frac{bK(\tau)}{Cr}\cos\theta + \frac{K(\tau)}{C^2r^2}\sin\theta\,\mu_3^{*\mathrm{s}}, \quad b' = \frac{\mu_2^*}{A} + \frac{aK(\tau)}{Cr}\cos\theta - \frac{K(\tau)}{C^2r^2}\cos\theta\,\mu_3^{*\mathrm{c}},\\
&r' = \frac{\mu_3^*}{C}, \quad \psi' = \frac{K(\tau)}{Cr}, \quad \theta' = 0, \quad \lambda' = -\frac{K(\tau)}{Cr}\cos\theta,\\
&\mu_1^* = \frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|}(M_1^0\cos\gamma + M_2^0\sin\gamma)\,d\gamma, \quad \mu_2^* = \frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|}(M_1^0\sin\gamma - M_2^0\cos\gamma)\,d\gamma,\\
&\mu_3^* = \frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|} M_3^0\,d\gamma, \quad \mu_3^{*\mathrm{s}} = \frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|} M_3^0\sin\alpha\,d\gamma,\\
&\mu_3^{*\mathrm{c}} = \frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|} M_3^0\cos\alpha\,d\gamma.
\end{aligned} \tag{2.7}$$

As has been mentioned, the dependence of the restoring torque on the slow time τ leads to the appearance of an additional term in system (2.4). However, this term disappears after averaging system (2.4), both in the non-resonant and resonant cases. As a result, we obtain systems similar to those of [1, 2], with the only difference that the restoring torque in the relations of (2.5) and (2.7) depends on the slow time, i.e., we have $K = K(\tau)$.

In what follows, we apply the described technique to specific cases of perturbed motion of a rigid body.

3. EXAMPLES

3.1. Linear dissipation. We will investigate the perturbed motion of Lagrange's top, with torques applied to the body by the environment being taken into account. This is the case, for example, for a medium the viscous properties of which slowly change due to changes in the density, temperature, and composition of the medium. We assume that the perturbation torques are linearly dissipative and, with reference to (1.3), have the form

$$M_1 = -\varepsilon^2 I_1(\tau)P, \quad M_2 = -\varepsilon^2 I_1(\tau)Q, \quad M_3 = -\varepsilon I_3(\tau)r, \tag{3.1}$$

where $I_1(\tau)$ and $I_2(\tau)$ are positive integrable functions defined on an interval $\tau \sim 1$. After a number of transformations, the solution of the first-approximation averaged system (2.5) for the perturbation torques of (3.1) can be represented as

$$\begin{aligned}
&\theta = \theta_0, \quad r(t) = r_0 \exp[F_3(\tau)], \quad \psi(\tau) = \psi_0 + \frac{1}{Cr_0}\int_0^\tau K(\tau^*)\exp[-F_3(\tau^*)]\,d\tau^*,\\
&a(\tau) = \exp[F_1(\tau)]\left[P_0\cos\beta + Q_0\sin\beta - \frac{K_0}{Cr_0}\sin\theta_0\sin(\beta+\varphi_0)\right],\\
&b(\tau) = \exp[F_1(\tau)]\left[P_0\sin\beta - Q_0\cos\beta + \frac{K_0}{Cr_0}\sin\theta_0\cos(\beta+\varphi_0)\right],\\
&F_1(\tau) = -\frac{1}{A}\int_0^\tau I_1(\tau^*)\,d\tau^*, \quad F_3(\tau) = -\frac{1}{C}\int_0^\tau I_3(\tau^*)\,d\tau^*,\\
&\beta = \frac{1}{Cr_0}\cos\theta_0\int_0^\tau K(\tau^*)\exp[-F_3(\tau^*)]\,d\tau^* = (\psi-\psi_0)\cos\theta_0.
\end{aligned}\tag{3.2}$$

By substituting the expressions of (3.2) for a, b, and r into the expressions of (2.1) and (1.3) for P, Q, p, and q we determine the desired variables

$$\begin{aligned}
&p = \exp[F_1(\tau)]\left[p_0\cos(\gamma-\beta) - q_0\sin(\gamma-\beta) + \frac{k_0}{Cr_0}\sin\theta_0\sin(\gamma-\beta-\varphi_0)\right] + \frac{k(\tau)}{Cr_0}\exp[-F_3(\tau)]\sin\theta_0\sin\varphi,\\
&q = \exp[F_1(\tau)]\left[p_0\sin(\gamma-\beta) + q_0\cos(\gamma-\beta) - \frac{k_0}{Cr_0}\sin\theta_0\cos(\gamma-\beta-\varphi_0)\right] + \frac{k(\tau)}{Cr_0}\exp[-F_5(\tau)]\sin\theta_0\cos\varphi,\\
&\gamma = \frac{r_0(C-A)}{A}\int_0^t \exp[F_3(\varepsilon t^*)]\,dt^*.
\end{aligned}\tag{3.3}$$

Note some qualitative features of this motion. According to (3.2), the nutation angle θ is constant. The magnitude of the axial angular velocity, r, monotonically (exponentially) decreases. In case the variables I_1 and I_2 are isolated from zero, the magnitudes of the slow variables a and b exponentially approach zero. The increment of the precession angle, $\psi-\psi_0$, depend on the product of the integrands $K(\tau)$ and $r^{-1}(\tau)$. The variable β changes in a similar manner, provided that $\theta_0 \neq \pm\pi/2$, and has a meaning of the oscillation phase. According to (3.3), the terms in the expressions for p and q accounted for by the initial values k_0, p_0, and q_0, exponentially decay. At the same time, these expressions involve also terms depending on the form of the restoring torque $k(\tau)$.

If $k = \text{const}$, $I_1 = \text{const}$, and $I_3 = \text{const}$, the expressions of (3.2) and (3.3) for r, ψ, a, b, p, and q coincide with the respective relations of [1]. The dependence of the restoring torque on the slow time complicates the expressions for the precession angle, ψ, slow variables a and b, and the equatorial components of the angular velocity, p and q.

Consider a particular case where the restoring torque slowly changes in time and has the form

$$k(\tau) = \varepsilon K(\tau) = \varepsilon(K_0 + \alpha_0\tau) = k_0 + \varepsilon\alpha_0\tau, \qquad \alpha_0 \neq 0, \quad \alpha_0 = \text{const}. \tag{3.4}$$

In system (3.2), the function $K(\tau)$ occurs in the equation for the precession angle, ψ, and in the expression for β; we have

$$\psi - \psi_0 = \frac{K_0}{Cr_0}\int_0^\tau \exp[-F_3(\tau^*)]\,d\tau^* + f_1, \quad f_1 = \frac{\alpha_0}{Cr_0}\int_0^\tau \tau^*\exp[-F_3(\tau^*)]\,d\tau^*, \qquad \beta = (\psi-\psi_0)\cos\theta_0. \tag{3.5}$$

In system (3.3), the expressions for p and q contain $k(\tau)$ in the second term; for p we have

$$p = \exp[F_1(\tau)]\left[p_0\cos(\gamma-\beta) - q_0\sin(\gamma-\beta) + \frac{k_0}{Cr_0}\sin\theta_0\sin(\gamma-\beta-\varphi_0)\right] + \frac{k_0}{Cr_0}\sin\theta_0\sin\varphi\exp[-F_3(\tau)] + f_2, \tag{3.6}$$

$$f_2 = \frac{\varepsilon\alpha_0\tau}{Cr_0}\sin\theta_0\sin\varphi\exp[-F_3(\tau)].$$

The expression for q has a similar form.

Note that the expressions of (3.5) and (3.6) for p, q, and ψ contain an exponentially increasing term involving the constant K_0, as is the case for the respective expressions of [1]. The difference is that the expressions of (3.5) and (3.6) involve additional terms, f_1 and f_2, respectively. The absolute value, $|f_1|$, of the additional term for the precession angle exponentially increases; the function f_1 is positive, equal to zero, and negative for $\alpha_0 > 0$, $\alpha_0 = 0$, and $\alpha_0 < 0$, respectively. In the expressions for p and q, the function f_2 has the form of a secular term. The case of $\alpha_0 = 0$ corresponds to $k = \text{const}$.

Let us briefly investigate the system of (2.5) in the case where

$$k(\tau) = \varepsilon K(\tau) = \varepsilon(K_0 + \alpha_0\sin\nu\tau), \qquad \alpha_0 \neq 0. \tag{3.7}$$

In the expressions of (3.2) and (3.3), the torque $K(\tau)$ occurs in the expressions for the precession angle, ψ, the argument β, and the projections of the angular velocity, p and q; for ψ and p, we have

$$\psi - \psi_0 = \frac{K_0}{Cr_0}\int_0^\tau \exp[-F_3(\tau^*)]\,d\tau^* + f_3,$$

$$p = \exp[F_1(\tau)]\left[p_0\cos(\gamma-\beta) - q_0\cos(\gamma-\beta) + \frac{k_0}{Cr_0}\sin\theta_0\sin(\gamma-\beta-\varphi_0)\right] + \frac{k_0}{Cr_0}\exp[-F_3(\tau)]\sin\theta_0\sin\varphi + f_4, \tag{3.8}$$

$$f_3 = \frac{\alpha_0}{Cr_0}\int_0^\tau \sin\nu\tau^*\exp[-F_3(\tau^*)]\,d\tau^*, \quad f_4 = \frac{\varepsilon\alpha_0}{Cr}\sin\nu\tau\sin\theta_0\sin\varphi.$$

The expression for q is similar to the expression of (3.8) for p.

Compare the expressions of (3.8) for p and ψ with the respective expressions of (3.5) and (3.6). A difference is observed only in the structure of the additional terms f_i ($i = 1, 2, 3, 4$). For $\alpha_0 = 0$ (or/and $\nu = 0$), the functions f_3 and f_4 vanish and the results correspond to the case of $k = \text{const}$.

3.2. Control of the equatorial component of the angular velocity. Consider the problem of bringing the top to the state of regular precession, in particular to the "sleeping" state. The small control torques are assumed to have the form

$$M_1 = -\varepsilon^2 h(\tau)\frac{p^*}{\sqrt{p^{*2}+q^{*2}}}, \quad M_2 = -\varepsilon^2 h(\tau)\frac{q^*}{\sqrt{p^{*2}+q^{*2}}}, \quad M_3 = \varepsilon u(\tau), \tag{3.9}$$

$$p^* = p - \frac{k(\tau)}{Cr}\sin\theta\sin\varphi, \quad q^* = q - \frac{k(\tau)}{Cr}\sin\theta\cos\varphi,$$

where $h(\tau)$ and $u(\tau)$ are prescribed integrable functions on an interval $\tau \sim 1$; $h(\tau) > 0$, $r \sim 1$. These control laws correspond to the time-optimal deceleration of the equatorial component of the angular velocity [5].

In accordance with (3.9), with reference to the expressions of (1.3) and (2.3) for p and q, the perturbation torques can be represented as follows:

$$M_1 = -\varepsilon^2 h(\tau)\frac{a\cos\gamma + b\sin\gamma}{\sqrt{a^2+b^2}}, \quad M_2 = -\varepsilon^2 h(\tau)\frac{a\sin\gamma - b\cos\gamma}{\sqrt{a^2+b^2}}, \quad M_3 = \varepsilon u(\tau). \tag{3.10}$$

Substitute (3.10) into (2.7) and then integrate the resulting equations to obtain the solution in the form

$$\theta = \theta_0, \quad r(\tau) = r_0 + \frac{1}{C}\int_0^\tau u(\tau^*)\,d\tau^*, \quad \psi(\tau) = \psi_0 + \frac{1}{C}\int_0^\tau \frac{K(\tau^*)}{r(\tau^*)}\,d\tau^*,$$

$$a(\tau) = F_4(\tau)\left[P_0\cos\chi + Q_0\sin\chi - \frac{K_0}{Cr_0}\sin\theta_0\sin(\chi+\varphi_0)\right], \tag{3.11}$$

$$b(\tau) = F_4(\tau)\left[P_0\sin\chi - Q_0\cos\chi + \frac{K_0}{Cr_0}\sin\theta_0\cos(\chi+\varphi_0)\right],$$

$$F_4(\tau) = \left[1 - \frac{1}{A\sqrt{a_0^2 + b_0^2}} \int_0^\tau h(\tau^*)\, d\tau^*\right], \quad \chi = \frac{\cos\theta_0}{C} \int_0^\tau \frac{K(\tau^*)}{r(\tau^*)}\, d\tau^*.$$

By substituting the expressions of (2.3) and (3.11) for P, Q, a, b, and r into the relations of (1.3) we determine the desired quantities

$$\begin{aligned} p &= F_4(\tau)\left[p_0\cos(\gamma-\chi) - q_0\sin(\gamma-\chi) + \frac{k_0}{Cr_0}\sin\theta_0\sin(\gamma-\chi-\varphi_0)\right] + \frac{k(\tau)}{Cr(\tau)}\sin\theta_0\sin\varphi, \\ q &= F_4(\tau)\left[p_0\sin(\gamma-\chi) + q_0\cos(\gamma-\chi) - \frac{k_0}{Cr_0}\sin\theta_0\cos(\gamma-\chi-\varphi_0)\right] + \frac{k(\tau)}{Cr(\tau)}\sin\theta_0\cos\varphi, \\ \gamma &= \frac{C-A}{A}\left[r_0 t + \frac{1}{C}\int_0^t \left(\int_0^{\tau^*} u(\tau_1)\, d\tau_1\right) d\tau^*\right], \qquad \tau = \varepsilon t. \end{aligned} \tag{3.12}$$

Thus we have obtained the solution of the system of (2.7) and (3.9) and have found expressions for the projections of the angular velocity in the case of the torque defined by (3.10). The nutation angle θ is constant. The quantity $|r(\tau)|$ increases if the parameter r_0 and the integral of the function $u(\tau)$ coincide in sign and decreases otherwise. The variables a and b are expressed as the product of a multiplier that assumes positive, negative, or zero values, depending on the integrand $h(\tau)$, by an oscillating multiplier. From (3.11) it follows that for $h(\tau) \geq h_0 > 0$, there exists

$$\tau_* \leq \tau_0 = \frac{A\sqrt{a_0^2 + b_0^2}}{h_0}$$

such that $a(\tau_*) = b(\tau_*) = 0$. If $K(\tau) = K_*$ and $r(\tau) = r_*$ for $\tau \geq \tau_*$, then the top performs a regular precession; in particular, it is "sleeping" for $\theta_0 = 0$ or $\theta_0 = \pi$. The increment of the precession angle, $\psi - \psi_0$, depends on the integral of the quotient of the restoring torque by the axial component of the angular velocity. This integral is positive if $K(\tau)$ and $r^{-1}(\tau)$ coincide in sign.

In accordance with (3.12), the components p and q of the angular velocity contain bounded oscillating terms, as well as a term determined by the restoring torque $k(\tau)$; the frequency of the oscillating terms depends on the variable $\gamma - \chi$.

The function $h(\tau)$ can be interpreted as a constraint on the control function, for example, in the problem of the deceleration of the equatorial component of the angular velocity by means of a constrained torque. In this case, M_1, M_2, and M_3 are control functions for the variables p, q, and r, respectively.

Consider the case where the restoring torque has the form

$$k(\tau) = \varepsilon K(\tau) = \varepsilon(K_0 + \alpha_0\tau) = k_0 + \varepsilon\alpha_0\tau, \qquad \alpha_0 \neq 0. \tag{3.13}$$

In this case, the precession angle, ψ, and the component p of the angular velocity are expressed as follows:

$$\begin{aligned} \psi - \psi_0 &= \frac{K_0}{C}\int_0^\tau \frac{d\tau^*}{r(\tau^*)} + \eta_1, \\ p &= F_4(\tau)\left[p_0\cos(\gamma-\chi) - q_0\sin(\gamma-\chi) + \frac{k_0}{Cr_0}\sin\theta_0\sin(\gamma-\chi-\varphi_0)\right] + \frac{k_0}{Cr(\tau)}\sin\theta_0\sin\varphi + \eta_2, \\ \eta_1 &= \frac{\alpha_0}{C}\int_0^\tau \frac{\tau^*}{r(\tau^*)}\, d\tau^*, \quad \eta_2 = \frac{\varepsilon\alpha_0\tau}{Cr(\tau)}\sin\theta_0\sin\varphi. \end{aligned} \tag{3.14}$$

The expression for the component q of the angular velocity is similar to that of (3.14) for the component p.

As was the case in [1], the expressions of (3.14) for p, q, and ψ involve terms that contain the constant k_0. However, unlike the expressions of [1], the expression of (3.14) for ψ contains an additional term η_1 and the expressions for p and q contain an additional term η_2.

Consider briefly the case where the restoring torque has the form

$$k(\tau) = \varepsilon K(\tau) = \varepsilon(K_0 + \alpha_0\sin\nu\tau), \qquad \alpha_0 \neq 0. \tag{3.15}$$

In this case, for ψ and p, we have the expressions

$$\begin{aligned} \psi - \psi_0 &= \frac{K_0}{C} \int_0^\tau \frac{d\tau^*}{r(\tau^*)} + \eta_3, \\ p &= F_4(\tau) \left[p_0 \cos(\gamma - \chi) - q_0 \sin(\gamma - \chi) + \frac{k_0}{C r_0} \sin\theta_0 \sin(\gamma - \chi - \varphi_0) \right] + \frac{k_0}{C r(\tau)} \sin\theta_0 \sin\varphi + \eta_4, \\ \eta_3 &= \frac{\alpha_0}{C} \int_0^\tau \frac{\sin \nu\tau^*}{r(\tau^*)} d\tau^*, \quad \eta_4 = \frac{\alpha_0}{C r(\tau)} \sin \nu\tau \sin\theta_0 \sin\varphi. \end{aligned} \tag{3.16}$$

The expression for q is similar to that of (3.16) for p.

The difference between the cases of (3.13) and (3.15) is in additional terms η_1, η_2, η_3, and η_4. Since $|\sin \nu\tau| < |\nu\tau|$, we have $|\nu_3| < |\nu_1|$ and $|\nu_4| < |\nu_2|$. Therefore, since the function $r(\tau)$ is bounded, the additional terms are also bounded.

If the resonant relation holds, i.e., $C/A = i/j$, where $i/j \le 2$, while i and j are positive coprime integers, one should utilize the scheme of (2.7) when averaging. In the previous examples, all integrals μ_i^* of (2.7) coincide with the respective integrals μ_i of (2.5). Therefore, in fact, there is no resonance and, hence, the obtained solution can be applied to describe the motion for any $C/A \ne 1$ ($C \le 2A$).

4. CONCLUSIONS

1. A new class of rotations of a dynamically symmetric rigid body about a fixed point has been investigated with unsteady restoring and perturbation torques being taken into account.

2. An averaging procedure has been developed for the resulting essentially nonlinear system both in non-resonant and resonant cases.

3. A number of specific problems of dynamics and control of rotation of a rigid body have been solved. The motions that have been considered are close to the regular precession in Lagrange's case. The obtained solutions have an independent value for applications.

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REFERENCES

[1] L. D. Akulenko, D. D. Leshchenko, and F. L. Chernousko, "Perturbed motions of a rigid body close to a regular precession," Izv. AN SSSR. MTT [Mechanics of Solids], No. 5, pp. 3–10, 1986.

[2] D. D. Leshchenko and S. N. Sallam, "Perturbed motions of a rigid body about a fixed point," Izv. AN SSSR. MTT [Mechanics of Solids], No. 5, pp. 16–23, 1990.

[3] N. N. Bogolyubov and Yu. A. Mitropol'skii, Asymptotic Methods in the Theory of Nonlinear Oscillations [in Russian], Nauka, Moscow, 1974.

[4] V. M. Volosov and B. I. Morgunov, Method of Averaging in the Theory of Nonlinear Oscillating Systems [in Russian], Izd-vo MGU, Moscow, 1971.

[5] L. D. Akulenko, Asymptotic Methods of Optimal Control [in Russian], Nauka, Moscow, 1987.

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