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см. с. 1-2, и Conclusions

CONTROL IN DETERMINISTIC SYSTEMS

Evolution of Rotations of a Rigid Body under the Action of Restoring and Control Moments

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Abstract—Perturbed rotations of a rigid body close to the regular precession in the Lagrangian case under the action of a restoring moment depending on slow time and nutation angle, as well as a perturbing moment slowly varying with time, are studied. The body is assumed to spin rapidly, and the restoring and perturbing moments are assumed to be small with a certain hierarchy of smallness of the components. A first approximation averaged system of equations of motion for an essentially nonlinear two-frequency system is obtained in both the nonresonance and resonance cases. Examples of motion of a body under the action of particular restoring, perturbing, and control moments of force are considered.

1. STATEMENT OF THE PROBLEM

Consider the motion of a dynamically symmetrical rigid body about a fixed point O under the action of a restoring moment depending on a slow time $\tau = \varepsilon t$ and a nutation angle θ as well as a perturbing moment slowly varying with time. The equations of motion have the form [1]

$$A\dot{p} + (C-A)qr = k(\tau, \theta)\sin\theta\cos\varphi + M_1,$$
$$A\dot{q} + (A-C)pr = -k(\tau, \theta)\sin\theta\sin\varphi + M_2,$$
$$C\dot{r} = M_3, \quad M_l = M_l(p, q, r, \psi, \theta, \varphi, \tau), \quad \tau = \varepsilon t \quad (l = 1, 2, 3), \tag{1.1}$$
$$\dot{\psi} = (p\sin\varphi + q\cos\varphi)\operatorname{cosec}\theta, \quad \dot{\theta} = p\cos\varphi - q\sin\varphi,$$
$$\dot{\varphi} = r - (p\sin\varphi + q\cos\varphi)\cot\theta.$$

Here, p, q, and r are the projections of the vector of angular velocity of the body onto the principal axes of inertia of the body originating at the point O. The values M_l are the projections of the vector of the perturbing moment onto the same axes. They depend on the slow time $\tau = \varepsilon t$ and are periodic functions of the Euler angles ψ, φ, and θ with periods 2π. Here, A is the equatorial and C the centroidal moments of inertia of the body about the point O, $A \neq C$. The body is assumed to be subjected to a restoring moment $k(\tau, \theta)$ slowly varying with time and 2π-periodically dependent on the nutation angle. In the absence of perturbations, when $M_l = 0$ and $k(\tau, \theta) = \text{const}$, Eqs. (1.1) correspond to the case of a Lagrangian gyroscope.

System (1.1) is examined under the following assumptions:

$$(p^2+q^2)^{1/2} \ll r, \quad Cr^2 \gg k, \quad |M_{1,2}| \ll k, \quad M_3 \sim k, \tag{1.2}$$

which mean that the direction of the angular velocity of the body is close to the axis of dynamical symmetry; the angular velocity of axial rotation is sufficiently large so that the kinetic energy of the body is much greater than the potential energy determined by the restoring moment; the two projections of the vector of the perturbing moment onto the principal axes of inertia of the body are small as compared to the restoring moment; and the third projection is of the same order as the restoring moment.

Inequalities (1.2) allow one to introduce the following relations:

$$p = \varepsilon P, \quad q = \varepsilon Q, \quad k(\tau, \theta) = \varepsilon K(\tau, \theta), \quad \tau = \varepsilon t,$$
$$M_{1,2} = \varepsilon^2 M^*_{1,2}(P, Q, r, \psi, \theta, \varphi, \tau), \tag{1.3}$$
$$M_3 = \varepsilon M^*_3(P, Q, r, \psi, \theta, \varphi, \tau).$$

The new normalized variables P and Q and the functions K and M^*_l $(l = 1, 2, 3)$, as well as the variables r, ψ, and θ and the parameters A and C, are assumed to be bounded values of order one as $\varepsilon \longrightarrow 0$; the angle of pure rotation $\varphi \sim \varepsilon^{-1}$.

Earlier [1, 2], rapid rotations of a rigid body close to the Lagrangian case under the action of a constant restoring moment $k = \text{const}$ [1] were considered. The case where the restoring moment depends on the nutation angle $k = k(\theta)$ and the perturbing moment also depends on the slow time $\tau = \varepsilon t$ was studied in [3].

Below, we examine a more general case where the restoring moment depends on both the nutation angle and the slow time $k = k(\tau, \theta)$; in particular, $k = k_0(\tau) + k_1(\tau)\cos\theta$. The perturbing moment is assumed to be slowly varying with time and is represented by functions of the form $M_l = M_l(p, q, r, \psi, \theta, \varphi, \tau)$.

It is proposed to investigate the asymptotic behavior of system (1.1) for small ε provided that conditions (1.2) and (1.3) are valid. For this purpose, we employ the method of averaging [4, 5] over the time interval of order ε^{-1}. The method of averaging enjoys wide applications in problems of the dynamics of rigid bodies. The simplifying assumptions (1.2) or (1.3) allow one to obtain in the general case a rather simple scheme of averaging and to investigate a number of examples.

2. CONSTRUCTION OF AVERAGED MOTION EQUATIONS

Let us make the change of variables (1.3) in system (1.1). Dividing both sides of the first two equations in (1.1) by ε, we obtain

$$A\dot{P} + (C-A)Qr = K(\tau, \theta)\sin\theta\cos\varphi + \varepsilon M_1^*,$$
$$A\dot{Q} + (A-C)Pr = -K(\tau, \theta)\sin\theta\sin\varphi + \varepsilon M_2^*,$$
$$C\dot{r} = \varepsilon M_3^*, \quad \dot{\psi} = \varepsilon(P\sin\varphi + Q\cos\varphi)\operatorname{cosec}\theta, \tag{2.1}$$
$$\dot{\varphi} = r - \varepsilon(P\sin\varphi + Q\cos\varphi)\cot\theta,$$
$$\dot{\theta} = \varepsilon(P\cos\varphi - Q\sin\varphi).$$

In terms of [4, 5], system (2.1) is two-frequency and essentially nonlinear, since the frequencies depend on a slow variable r.

We start with considering the first approximation system and set $\varepsilon = 0$ in (2.1). From the last four equations, we find

$$r = r_0, \quad \psi = \psi_0, \quad \theta = \theta_0, \qquad \varphi = r_0 t + \varphi_0, \quad K_0 = K(\tau_0, \theta_0). \tag{2.2}$$

Here, r_0, ψ_0, θ_0, φ_0, and τ_0 are constants equal to the initial values of the variables at $t = 0$. Substituting equalities (2.2) into the first two equations in (2.1) for $\varepsilon = 0$ and integrating the obtained system of linear equations for P and Q, we obtain

$$P = a\cos\gamma + b\sin\gamma + KC^{-1}r^{-1}\sin\theta\sin\varphi,$$
$$Q = a\sin\gamma - b\cos\gamma + KC^{-1}r^{-1}\sin\theta\cos\varphi,$$
$$a = P_0 - K_0C^{-1}r_0^{-1}\sin\theta_0\sin\varphi_0, \tag{2.3}$$
$$b = -Q_0 + K_0C^{-1}r_0^{-1}\sin\theta_0\cos\varphi_0,$$
$$\dot{\gamma} = n, \quad \gamma(0) = 0, \quad n = (C-A)A^{-1}r \neq 0,$$
$$|n/r| \leq 1, \quad \alpha = \varphi + \gamma.$$

Here, a and b are the osculating variables of the Van der Pol type introduced instead of (1.3) and the variable γ can be treated as an oscillation phase.

Consider system (2.1) for $\varepsilon \neq 0$ and relations (2.3) as the formulas of the change of variables (containing the variable γ) that define the passage from the variables P, Q to the variables a, b and back. Using these formulas, we pass in system (2.1) from the variables $P, Q, r, \psi, \theta, \varphi$, and τ to the new variables $a, b, r, \psi, \theta, \alpha, \gamma$, and τ. Note that the phases φ, α, and γ are related by a finite relation, which turns out to be more convenient for further investigation of a standard system with two rotating phases γ and α. Performing some transformations, we obtain a system of the form

$$\dot{a} = \varepsilon A^{-1}(M_1^0\cos\gamma + M_2^0\sin\gamma)$$
$$+ \varepsilon K(\tau, \theta)C^{-2}r^{-2}M_3^0\sin\theta\sin\theta$$
$$-\varepsilon K(\tau, \theta)C^{-1}r^{-1}\cos\theta(b - K(\tau, \theta)C^{-1}r^{-1}\sin\theta\cos\alpha)$$
$$-\varepsilon C^{-1}r^{-1}\sin\theta\sin\alpha\left[\frac{\partial K}{\partial\theta}(a\cos\alpha + b\sin\alpha) + \frac{\partial K}{\partial\tau}\right],$$
$$\dot{b} = \varepsilon A^{-1}(M_1^0\sin\gamma - M_2^0\cos\gamma)$$
$$-\varepsilon K(\tau, \theta)C^{-2}r^{-2}M_3^0\sin\theta\cos\alpha$$
$$+\varepsilon K(\tau, \theta)C^{-1}r^{-1}\cos\theta(a + K(\tau, \theta)C^{-1}r^{-1}\sin\theta\sin\alpha) \tag{2.4}$$
$$+\varepsilon C^{-1}r^{-1}\sin\theta\cos\alpha\left[\frac{\partial K}{\partial\theta}(a\cos\alpha + b\sin\alpha) + \frac{\partial K}{\partial\tau}\right],$$
$$\dot{r} = \varepsilon C^{-1}M_3^0, \quad \dot{\theta} = \varepsilon(a\cos\alpha + b\sin\alpha),$$
$$\dot{\psi} = \varepsilon(a\sin\alpha - b\cos\alpha)\operatorname{cosec}\theta + \varepsilon K(\tau, \theta)C^{-1}r^{-1},$$
$$\dot{\alpha} = CA^{-1}r - \varepsilon(a\sin\alpha - b\cos\alpha)\cot\theta$$
$$-\varepsilon K(\tau, \theta)C^{-1}r^{-1}\cos\theta, \quad \dot{\gamma} = (C-A)A^{-1}r,$$
$$M_l^0(a, b, r, \psi, \theta, \alpha, \gamma, \tau)$$
$$= M_l^*(P, Q, r, \psi, \theta, \varphi, \tau) \quad (l = 1, 2, 3).$$

Note that, when $K = \text{const}$ and M_l is independent of τ, system (2.4) coincides with the corresponding system that was investigated in [1].

Let us study the possibility of applying the method of averaging to system (2.4). This system contains slow variables a, b, r, ψ, θ, and τ and fast variables, namely, the phases α and γ. The dependence of the restoring moment on the slow variable τ and on the nutation angle θ causes the appearance of terms containing the derivatives $\frac{\partial K}{\partial \tau}$ and $\frac{\partial K}{\partial \theta}$ in the first two equations of system (2.4). If the perturbing moments depend on time t, then the method of averaging can hardly be applied, because the system is essentially nonlinear. Consider a

simpler case of the dependence of the perturbing moments on the slow time $\tau = \varepsilon t$.

The moments M_l^* are periodic in φ with period 2π; therefore, according to (2.3), the functions M_l^0 are 2π-periodic functions of α and γ. In this case, system (2.4) contains two rotating phases α and γ and the corresponding frequencies $\omega_\alpha = CA^{-1}r$ and $\omega_\gamma = (C-A)A^{-1}r$ are varying and depend on the slow variable r. When averaging system (2.4), two cases should be distinguished: the nonresonance case, namely, when the frequencies ω_γ and ω_α are incommensurable (C/A is an irrational number), and the resonance case, namely, when these frequencies are commensurable ($C/A = i/j$, $i/j \le 2$, i, j are positive coprime integers). Since the ratio of the frequencies is constant, $\omega_\gamma/\omega_\alpha = 1 - AC^{-1}$, as a result of introducing the variable γ, the averaging of nonlinear system (2.4) is equivalent to the averaging of a quasilinear system with constant frequencies.

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In the nonresonance case ($C/A \ne i/j$), we obtain the first approximation averaged system by independently averaging the right-hand sides of system (2.4) with respect to both fast variables α and γ. Making a change of the argument $\tau = \varepsilon t$ and dividing both sides of the equations by ε, we obtain

$$
\begin{aligned}
a' &= A^{-1}\mu_1 - 1/2C^{-1}r^{-1}b\sin\theta\frac{\partial K}{\partial\theta} \\
&\quad - bK(\tau,\theta)C^{-1}r^{-1}\cos\theta + K(\tau,\theta)C^{-2}r^{-2}\sin\theta\mu_3^s, \\
b' &= A^{-1}\mu_2 + 1/2C^{-1}r^{-1}a\sin\theta\frac{\partial K}{\partial\theta} \\
&\quad + aK(\tau,\theta)C^{-1}r^{-1}\cos\theta - K(\tau,\theta)C^{-2}r^{-2}\sin\theta\mu_3^c, \\
r' &= C^{-1}\mu_3, \quad \psi' = K(\tau,\theta)C^{-1}r^{-1}, \quad \theta' = 0, \\
\mu_1 &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}(M_1^0\cos\gamma + M_2^0\sin\gamma)\,d\alpha\,d\gamma, \\
\mu_2 &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}(M_1^0\sin\gamma - M_2^0\cos\gamma)\,d\alpha\,d\gamma, \\
\mu_3 &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}M_3^0\,d\alpha\,d\gamma, \\
\mu_3^s &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}M_3^0\sin\alpha\,d\alpha\,d\gamma, \\
\mu_3^c &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}M_3^0\cos\alpha\,d\alpha\,d\gamma,
\end{aligned}
\tag{2.5}
$$

where $'$ denotes the derivative with respect to τ ($\tau \sim 1$, $t \sim \varepsilon^{-1}$).

In the resonance case, system (2.4) is a single-frequency system. Instead of α, we introduce a slow variable λ, namely, a linear combination of phases with integer coefficients

$$
\lambda = \alpha - i(i-j)^{-1}\gamma, \quad i/j \ne 1, \quad i/j \le 2, \quad i, j > 0. \tag{2.6}
$$

Then, Eqs. (2.4) take the form of a standard system with a rotating phase γ whose right-hand sides are periodic in γ with period $2|i-j|\pi$. We construct the first approximation system by averaging the right-hand sides of system (2.4) over the indicated period of variation of the argument γ. Making the change $\tau = \varepsilon t$, we reduce the system to the form

$$
\begin{aligned}
a' &= A^{-1}\mu_1^* - 1/2C^{-1}r^{-1}b\sin\theta\frac{\partial K}{\partial\theta} \\
&\quad - bK(\tau,\theta)C^{-1}r^{-1}\cos\theta + K(\tau,\theta)C^{-2}r^{-2}\sin\theta\mu_3^{*s}, \\
b' &= A^{-1}\mu_2^* + 1/2C^{-1}r^{-1}a\sin\theta\frac{\partial K}{\partial\theta} \\
&\quad + aK(\tau,\theta)C^{-1}r^{-1}\cos\theta - K(\tau,\theta)C^{-2}r^{-2}\sin\theta\mu_3^{*c}, \\
r' &= C^{-1}\mu_3^*, \quad \psi' = K(\tau,\theta)C^{-1}r^{-1}, \quad \theta' = 0, \\
\lambda' &= -K(\tau,\theta)C^{-1}r^{-1}\cos\theta, \\
\mu_1^* &= \frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|}(M_1^0\cos\gamma + M_2^0\sin\gamma)\,d\gamma, \\
\mu_2^* &= \frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|}(M_1^0\sin\gamma + M_2^0\cos\gamma)\,d\gamma, \\
\mu_3^* &= \frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|}M_3^0\,d\gamma, \\
\mu_3^{*s} &= \frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|}M_3^0\sin\alpha\,d\gamma, \\
\mu_3^{*c} &= \frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|}M_3^0\cos\alpha\,d\gamma.
\end{aligned}
\tag{2.7}
$$

The dependence of the restoring moment on the slow time τ and on the nutation angle θ caused the appearance of a term containing two derivatives in system (2.4). However, when averaging this system in either the nonresonance (2.5) or resonance (2.7) case, the derivative $\frac{\partial K}{\partial\tau}$ vanishes. As a result, we obtain systems containing the restoring moment $K(\tau, \theta)$ and the derivative $\frac{\partial K}{\partial\theta}$ similar to those of [2]. The only difference is that, in (2.5) and (2.7), the restoring moment and the derivative of the restoring moment with respect to the nutation angle depend on the slow time $\tau = \varepsilon t$.

Below, we investigate some particular cases of perturbed motion of a rigid body with the help of the above-presented methods.

3. EXAMPLES

3.1. Case of linear dissipation. Let us study the Lagrange perturbed motion taking into the account the moments that affect the rigid body from the exterior. A case in point is an environment with slowly varying properties of viscosity as a consequence of a change in density, temperature, and environment composition. We assume that the perturbing moments are linear dissipative and, with due regard for (1.3), take the form

$$M_1 = -\varepsilon^2 I_1(\tau)P, \quad M_2 = -\varepsilon^2 I_1(\tau)Q, \quad M_3 = -\varepsilon I_3(\tau)r. \tag{3.1}$$

Here, $I_1(\tau)$ and $I_3(\tau)$ are positive integrable functions defined for $\tau \sim 1$. Performing a number of transformations, one obtains the solution to the first approximation averaged system of equations (2.5) for perturbing moments (3.1) in the form

$$\begin{gathered}
\theta = \theta_0, \quad r(\tau) = r_0 \exp[F_3(\tau)], \\
\psi(\tau) = \psi_0 + C^{-1} r_0^{-1} \int_0^\tau K(\tau^*, \theta) \exp[-F_3(\tau^*)] d\tau^*, \\
a(\tau) = \exp[F_1(\tau)] \\
\times [P_0 \cos\beta + Q_0 \sin\beta - K_0 C^{-1} r_0^{-1} \sin\theta_0 \sin(\beta + \varphi_0)], \\
b(\tau) = \exp[F_1(\tau)] \\
\times [P_0 \sin\beta - Q_0 \cos\beta + K_0 C^{-1} r_0^{-1} \sin\theta_0 \cos(\beta + \varphi_0)], \\
F_1(\tau) = -A^{-1} \int_0^\tau I_1(\tau^*) d\tau^*, \\
F_3(\tau) = -C^{-1} \int_0^\tau I_3(\tau^*) d\tau^*, \\
\beta = C^{-1} r_0^{-1} \int_0^\tau \exp[-F_3(\tau^*)] \\
\times \left[K(\tau^*, \theta)\cos\theta_0 + 1/2 \sin\theta_0 \frac{\partial K}{\partial \theta}\right] d\tau^*.
\end{gathered} \tag{3.2}$$

Substituting the expressions for a, b, and r from (3.2) into relations (2.3) and (1.3) for P, Q, p, and q, we determine the desired variables

$$\begin{gathered}
p = \exp[F_1(\tau)][p_0 \cos(\gamma - \beta) - q_0 \sin(\gamma - \beta) \\
+ k_0 C^{-1} r_0^{-1} \sin\theta_0 \sin(\gamma - \beta - \varphi_0)] \\
+ k(\tau, \theta) C^{-1} r_0^{-1} \exp[-F_3(\tau)] \sin\theta_0 \sin\varphi, \\
q = \exp[F_1(\tau)][p_0 \sin(\gamma - \beta) + q_0 \cos(\gamma - \beta) \\
- k_0 C^{-1} r_0^{-1} \sin\theta_0 \cos(\gamma - \beta - \varphi_0)] \\
+ k(\tau, \theta) C^{-1} r_0^{-1} \exp[-F_3(\tau)] \sin\theta_0 \cos\varphi, \\
\gamma = A^{-1} r_0 (C - A) \int_0^t \exp[F_3(\varepsilon t')] dt'.
\end{gathered} \tag{3.3}$$

Note some qualitative features of the motion in this case. By (3.2), the nutation angle θ is constant. The absolute value of the axial rotation velocity r exponentially decreases. The values of the slow variables a and b exponentially vanish (in the case where I_1 and I_3 are separated from zero). The increase in the precession angle $\psi - \psi_0$ depends on the product of the integrands $K(\tau, \theta)$ and $r^{-1}(\tau)$. According to (3.3), the terms of the projections p and q that are determined by the initial values k_0, p_0, and q_0 exponentially decay. At the same time, the projections p and q contain some terms that depend on the form of the restoring moment $k(\tau, \theta)$.

In the case where k = const, I_1 = const, and I_3 = const, expressions (3.2) and (3.3) for r, ψ, a, b, p, and q coincide with the corresponding formulas in [1]. The dependence of the restoring moment on the slow time resulted in some complications as compared with [1] of the expressions for the precession angle ψ, slow variables a and b, and the equatorial components p and q of the vector of angular velocity.

As an example of the restoring moment that depends on the nutation angle and slowly varies with time, consider a rigid body with a spring attached to it at a point N, whose end L is fixed (see figure) [2]. The body is acted upon by the gravity force mg and the elastic force **F** whose magnitude is proportional to the strain of the spring $F = \delta(s - s_0)$. Here, δ is the rigidity coefficient of the spring and s_0 is the length of unstrained spring. In this case, the restoring moment has the form

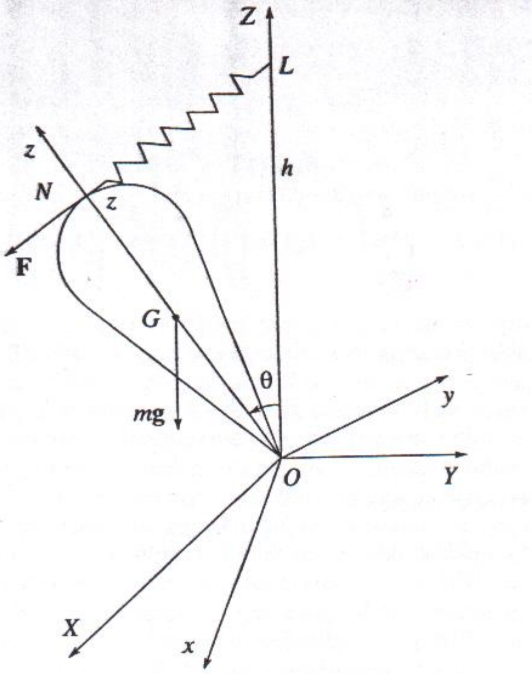
$$k(\tau, \theta) = \varepsilon K^*(\tau, \theta) = \varepsilon(K(\theta) + \xi\tau),$$

$$K(\theta) = mgl + \delta hz[1 - s_0(h^2 - z^2 - 2hz\cos\theta)^{-1/2}], \tag{3.4}$$

$$ON = z, \quad OL = h, \quad LN = s = s(\theta), \quad OC = l.$$

In system (3.2), the function $K^*(\tau, \theta)$ appears in the equations for the precession angle ψ and for the argument β:

$$\psi - \psi_0 = C^{-1} r_0^{-1} K(\theta) \int_0^\tau \exp[-F_3(\tau^*)] d\tau^* + f_1,$$



Mechanical model of the "Lagrangian gyroscope" with the restoring moment of forces depending on the nutation angle θ.

$$f_1 = \xi C^{-1} r_0^{-1} \int_0^{\tau} \tau^* \exp[-F_3(\tau^*)] d\tau^*, \tag{3.5}$$

$$\beta = C^{-1} r_0^{-1} [K(\theta)\cos\theta_0 + 1/2\delta h^2 z^2 s_0 \sin^2\theta_0 (h^2$$

$$+ z^2 - 2hz\cos\theta_0)^{-3/2}] \int_0^{\tau} \exp[-F_3(\tau^*)] d\tau^* + f_1 \cos\theta_0.$$

In system (3.3), the second terms of the projections p and q contain $k(\tau, \theta)$; thus, the expression for p has the form

$$p = \exp[F_1(\tau)][p_0\cos(\gamma - \beta) - q_0\sin(\gamma - \beta)$$

$$+ k_0 C^{-1} r_0^{-1} \sin\theta_0 \sin(\gamma - \beta - \varphi_0)] \tag{3.6}$$

$$+ k(\tau, \theta) C^{-1} r_0^{-1} \sin\theta_0 \sin\varphi \exp[-F_3(\tau)] + f_2,$$

$$f_2 = \varepsilon\xi C^{-1} r_0^{-1} \sin\theta_0 \sin\varphi \exp[-F_3(\tau)].$$

One obtains a similar expression for the variable q.

3.2. Control of the equatorial component of the vector of angular velocity. Consider the problem of bringing a gyroscope into the state of regular precession, in particular, into the "sleeping" state. Small control moments are assumed to have the form

$$M_1 = -\varepsilon^2 h(\tau) \frac{p^*}{(p^{*2} + q^{*2})^{1/2}},$$

$$M_2 = -\varepsilon^2 h(\tau) \frac{q^*}{(p^{*2} + q^{*2})^{1/2}},$$

$$M_3 = \varepsilon u(\tau), \tag{3.7}$$

$$p^* = p - k(\tau, \theta) C^{-1} r^{-1} \sin\theta \sin\varphi,$$

$$q^* = q - k(\tau, \theta) C^{-1} r^{-1} \sin\theta \cos\varphi.$$

Here, $h(\tau)$ and $u(\tau)$ are given integrable functions defined for $\tau \sim 1$; $h(\tau) > 0$. These control laws correspond to the time-optimal suppression of the equatorial component of the vector of angular velocity [6] (bringing it into the mode of regular precession).

In view of relations (1.3) and (2.3) for p and q, according to (3.7), the control moments have the form

$$M_1 = -\varepsilon^2 h(\tau) \frac{a\cos\gamma + b\sin\gamma}{(a^2 + b^2)^{1/2}},$$

$$M_2 = -\varepsilon^2 h(\tau) \frac{a\sin\gamma - b\cos\gamma}{(a^2 + b^2)^{1/2}}, \quad M_3 = \varepsilon u(\tau). \tag{3.8}$$

Substituting the control moments (3.8) into (2.5) and performing the integration, we obtain a solution of the form

$$\theta = \theta_0, \quad r(\tau) = r_0 + C^{-1} \int_0^{\tau} u(\tau^*) d\tau^*,$$

$$\psi(\tau) = \psi_0 + C^{-1} \int_0^{\tau} K(\tau^*, \theta) r^{-1}(\tau^*) d\tau^*,$$

$$a(\tau) = F_4(\tau)$$

$$\times [P_0\cos\chi + Q_0\sin\chi - K_0 C^{-1} r_0^{-1} \sin\theta_0 \sin(\chi + \varphi_0)],$$

$$b(\tau) = F_4(\tau) \tag{3.9}$$

$$\times [P_0\sin\chi - Q_0\cos\chi + K_0 C^{-1} r_0^{-1} \sin\theta_0 \cos(\chi + \varphi_0)],$$

$$F_4(\tau) = 1 - A^{-1}(a_0^2 + b_0^2)^{-1/2} \int_0^{\tau} h(\tau^*) d\tau^*,$$

$$\chi = C^{-1} \int_0^{\tau} \left[K(\tau^*, \theta)\cos\theta_0 + 1/2\sin\theta_0 \frac{\partial K}{\partial \theta}\right] r^{-1}(\tau^*) d\tau^*.$$

Substituting the expressions for P, Q, a, b, and r from (2.3) and (3.9) into relations (1.3), we determine the desired values

$$p = F_4(\tau)[p_0\cos(\gamma - \chi) - q_0\sin(\gamma - \chi)$$

$$+ k_0 C^{-1} r_0^{-1} \sin\theta_0 \sin(\gamma - \chi - \varphi_0)]$$

$$+k(\tau,\theta)C^{-1}r^{-1}(\tau)\sin\theta_0\sin\varphi,$$

$$q = F_4(\tau)[p_0\sin(\gamma-\chi)+q_0\cos(\gamma-\chi) \quad (3.10)$$
$$-k_0C^{-1}r_0^{-1}\sin\theta_0\cos(\gamma-\chi-\varphi_0)]$$
$$+k(\tau,\theta)C^{-1}r^{-1}(\tau)\sin\theta_0\cos\varphi,$$

$$\gamma = A^{-1}(C-A)\left[r_0t+C^{-1}\int_0^t\left(\int_0^{t^*}u(\tau_1)d\tau_1\right)dt^*\right], \quad \tau = \varepsilon t.$$

Thus, we have obtained the solutions to system (2.5), (3.7) in the case of moment (3.8) and found the expressions for the projections of the vector of angular velocity. The nutation angle θ is constant. The value $|r(\tau)|$ increases if the parameter r_0 has the same sign as the integral of the function $u(\tau)$ and decreases, otherwise. The variables a and b are the products of a factor that takes positive or negative values or is equal to zero depending on the integrand $h(\tau)$ and an oscillating factor. The increase in the precession angle $\psi-\psi_0$ is determined by the integral of the ratio of the restoring moment to the axial rotation velocity; it is positive in the case where $K(\tau,\theta)$ has the same sign as $r^{-1}(\tau)$.

According to (3.10), the components p and q of the vector of angular velocity contain bounded oscillating terms whose oscillation frequency is determined by the value $\gamma-\chi$ and a term determined by the restoring moment $k(\tau,\theta)$.

The function $h(\tau)$ can be treated as a constraint on the control action. Such interpretation allows one, for example, to solve the problem of suppressing the equatorial component by means of a bounded moment of forces, where $M_{1,2}$ is a control for p and q and M_3 is a control for r.

3.3. Axisymmetric body entry into the atmosphere. Consider the case where the restoring moment has the form

$$k(\tau,\theta) = \varepsilon K^*(\tau,\theta)$$
$$= \varepsilon(K(\theta)+\xi\sin\nu\tau) = k^*(\theta)+\varepsilon\xi\sin\nu\tau, \quad (3.11)$$
$$K(\theta) = A(\mu+2\eta\cos\theta).$$

Here, μ and η are constant coefficients with no constraints imposed on their signs. Such problems arise for uncontrolled spatial motion of a body in the atmosphere [7].

The expressions for the precession angle ψ, argument χ, and projections p and q of the vector of angular velocity take the form

$$\psi-\psi_0 = C^{-1}K(\theta_0)\int_0^\tau r^{-1}(\tau^*)d\tau^*+\eta_1,$$

$$\chi = C^{-1}[K(\theta_0)\cos\theta_0+A\eta\sin^2\theta_0]$$
$$\times\int_0^\tau r^{-1}(\tau^*)d\tau^*+\eta_1\cos\theta_0,$$

$$p = F_4(\tau)[p_0\cos(\gamma-\chi)-q_0\sin(\gamma-\chi) \quad (3.12)$$
$$+k^*(\theta_0)C^{-1}r_0^{-1}\sin\theta_0\sin(\gamma-\chi-\varphi_0)]$$
$$+k^*(\theta_0)C^{-1}r^{-1}(\tau)\sin\theta_0\sin\varphi+\eta_2,$$

$$\eta_1 = \xi C^{-1}\int_0^\tau \sin\nu\tau^* r^{-1}(\tau^*)d\tau^*,$$

$$\eta_2 = \varepsilon\xi\sin\nu\tau C^{-1}r^{-1}(\tau)\sin\theta_0\sin\varphi.$$

One obtains a similar formula for q.

As in [2], expressions (3.12) for p, q, and ψ contain terms involving $k^*(\theta_0)$. The difference is that they contain additional terms η_2 and η_1, respectively. Since the function $r(\tau)$ is bounded, the additional terms are also bounded and $|\sin\nu\tau|<|\nu\tau|$.

If the resonance relation $C/A = i/j$ ($i/j \le 2$, i, j are positive coprime integers) holds, then the system should be averaged according to scheme (2.9). In the examples given above, all integrals μ_l^* from (2.9) coincide with the corresponding integrals μ_l from (2.7). That is why there is actually no resonance and the obtained solution is good for describing the motion for any ratio $C/A \ne 1$.

CONCLUSIONS

(1) A new class of rotations of a dynamically symmetric rigid body about a fixed point with account for a nonstationary perturbing moment, as well as for a perturbing moment that slowly varies with time and depends on the nutation angle, is studied. This class is the widest among those known in the literature.

(2) A procedure for averaging the obtained essentially nonlinear two-frequency system is developed in both the nonresonance and resonance cases.

(3) Particular problems of the dynamics and control of rotations of a rigid body close to the regular precession in the Lagrangian case, which are of independent importance for applications, are solved.

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