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Problems of Evolution of Rotations of a Rigid Body under the Action of Perturbing Moments

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Abstract. The rotatory motion of a nearly dynamically spherical rigid body, which contains a viscoelastic element, is considered. This element is simulated by a moving mass, connected by a spring and damper to the point, situated on one of a principal axis of inertia. The small parameters caused by the proximity of moments of inertia and the presence of moving mass are considered to be of the same order. The spherical coordinates defining the position of the angular velocity vector are introduced. The system of differential equations is obtained and investigated, the special cases of motion are considered. The authors investigate perturbed rotational motions of a rigid body, similar to the regular precession in the Lagrange case, under the action of the moment that is slowly changing in time and the restoring moment depending on the angle of nutation. In two problems it is assumed that the angular velocity of the body is large and its direction is close to the axes of dynamic symmetry. In the first problem it is assumed that two projections of the vector of the perturbing moment onto principal axes of inertia of the body are small as compared to the restoring moment, while the third one is of the same order of the magnitude as the moment in question. In the second problem it is assumed that the perturbing moments are small as compared to the restoring one. Averaged systems of equations of motion are obtained and investigated in the first and the second approximations. Examples are considered.

Key words: evolution, rigid body, averaging method, rotation.

1. Introduction

The authors investigated some new problems of the motion of a rigid body about a fixed point under the action of the perturbing moments of different physical nature.

This paper treats problems of motion about the centre of inertia of a nearly dynamically spherical rigid body carrying a point mass connected to the body by the spring and the damper. It is assumed that the frequency for the main body is much lower that one for the point mass. Using the conservation of the moment of momentum of the system, a relation is developed for the angular velocity of the main body. The motion of a dynamically symmetric rigid body with a point mass m, attached to the point O_1 on its axes by means of the spring with the stiffness c

and the damping coefficient δ, is considered. The system of differential equations is obtained and investigated. It is similar to the obtained one in [2]. This system has the first integral. We qualitatively investigate the phase plane of the system. The critical points of the system are determined. The phase portraits of the system are constructed numerically, the phase curves describe oscillations and rotations.

We investigate perturbed rotatory motions of a rigid body, that are close to the regular precession in the Lagrange case, under the action of the moment of forces that is slowly changing in time and the restoring moment which depends on the nutation angle. It is assumed in unit 2 that: the angular velocity of the body is large; restoring and perturbing moments are small with definite hierarchy of smallness of components. The averaged system of equations of motion is obtained in the first approximation for the essentially nonlinear two-frequency system in nonresonant and resonant cases. Example is considered. The qualitative distinctions of motion are noted. In the second problem in unit 3 it is assumed that the angular velocity of the body is fairly large, its direction is close to the dynamic axis of symmetry of the body and the perturbing moments are small compared with the restoring moment. The small parameter is introduced by a special way and the method of averaging is used. The averaged equations of motion are obtained in the first and the second approximations. For the motion under the action of the resistance moment, applied by the medium, we have found out the evolution of the precession and nutation angles. The new class of motions of the axially symmetric body with allowance for the nonstationary perturbing moments is investigated. In [3, 6, 7] the perturbing moments were stationary.

2. The Motion of a Rigid Body Containing a Viscoelastic Element

We investigate the motion of a nearly dynamically spherical rigid body relative to the centre of inertia. In a point O_1, situated on one of the principal axes of inertia, the moving mass is attached by the spring and the damper. We locate the origin of the Cartesian coordinate system associated with the rigid body at the centre of inertia (point O) of the body and direct the basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ by the principal axis of inertia so, that the basis vector $\mathbf{e}_3$ defines the axis, on which the point O_1 is placed. Then the radius vector of the point O_1 is $\boldsymbol{\rho} = \rho\mathbf{e}_3$, and we consider $\rho > 0$ without loss of generality.

The scheme stated in [1] is utilized in the deduction of equations of motion. The following inequalities are assumed to be satisfied

$$\Omega^2 \gg \lambda\omega \gg \omega^2, \quad (\omega \equiv |\boldsymbol{\omega}| \sim 1). \tag{1}$$

Under the assumptions (1) the natural oscillations of the point mass m can be neglected and forced motions are taken into account. We shall look for the forced motion in the form of decomposition with the respect to degrees of Ω^{-2}. The equation of motion of the rigid body with the inertia tensor J_0^* can be written down in the following vector form

$$J_0^*\dot{\boldsymbol{\omega}} + (\boldsymbol{\omega} \times J_0^*\boldsymbol{\omega}) = \boldsymbol{\Phi}(\boldsymbol{\omega}) + O(\Omega^{-4}, \lambda^2\Omega^{-6}). \tag{2}$$

In (1) $\Omega^2 = c/m$, $\lambda = \delta/m$, where c is the rigidity of the elastic coupling of combination of the moving mass with the point of the body, δ is the coefficient of viscous friction. The inertia tensor J_0^* corresponds to the body with moving mass combined with O_1. The vector function $\boldsymbol{\Phi}$ contains terms of order Ω^{-2}, $\lambda\Omega^{-4}$ and is considered to be a polynomial that contains the fourth and the fifth powers of vector $\boldsymbol{\omega}$.

Suppose that the body principal central moments of inertia are nearly the same and can be represented in the form

$$J_{0_1}^* = J_0 + \varepsilon A', \quad J_{0_2}^* = J_0 + \varepsilon B', \quad J_{0_3}^* = J_0 + \varepsilon C', \tag{3}$$

where $0 < \varepsilon \ll 1$ is a small parameter. According to (1), Ω^{-2}, $\lambda\Omega^{-4}$ are small parameters in equations of motion (2). We assume that they are of the same order of smallness as the gyroscopic moments, i.e. $\Omega^{-2} \sim \varepsilon$, $\lambda\Omega^{-4} \sim \varepsilon$. Then, neglecting by small terms of the order two and higher, we obtain the equation of motion in the scalar form:

$$\begin{aligned}
\dot{p} &= \frac{\varepsilon}{J_0} qr[B' - C' + \rho^2 m(p^2 + q^2 + r^2)], \\
\dot{q} &= -\frac{\varepsilon}{J_0} pr[A' - C' + \rho^2 m(p^2 + q^2 + r^2)], \\
\dot{r} &= -\frac{\varepsilon}{J_0} pq(B' - A').
\end{aligned} \tag{4}$$

Here p, q, r are projections of vector $\boldsymbol{\omega}$ on the axes $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$.

By multiplying the three equations in (4) by p, q, r respectively and adding them, we obtain that the first integral of the system is

$$\omega^2 = p^2 + q^2 + r^2 = \omega_0^2 = \text{const}. \tag{5}$$

It allows us to introduce the angles Θ, φ, defining the orientation of vector $\boldsymbol{\omega}$ relative to the rigid body, as follows

$$p = \omega_0 \cos\varphi \sin\Theta, \quad q = \omega_0 \sin\varphi \sin\Theta, \quad r = \omega_0 \cos\Theta, \tag{6}$$

where $0 \le \Theta \le \pi$, $0 \le \varphi < 2\pi$.

Then we take Θ, φ as new variables in equations (4) and introduce the slow time

$$\tau = \varepsilon\omega_0 \frac{B' - A'}{J_0} t.$$

Solving obtained equations with respect to derivatives φ', Θ' by the slow time we find

$$\begin{aligned}
\Theta' &= \sin\Theta \sin\varphi \cos\varphi, \\
\varphi' &= \cos\Theta(\mu - \sin^2\varphi),
\end{aligned} \tag{7}$$

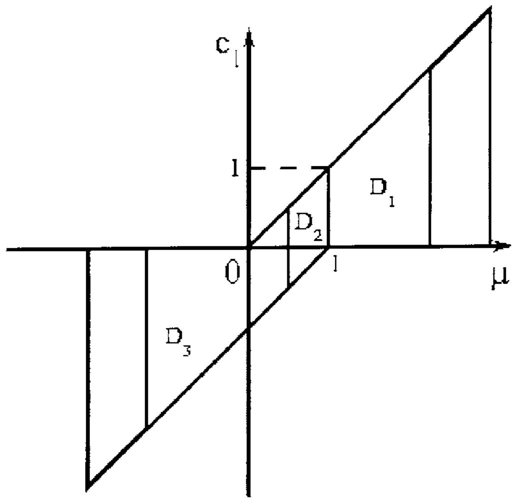

Figure 1. The domain $D = D_1 \cup D_2 \cup D_3$.

where

$$\mu = \frac{A' - C' + \rho^2 m\omega_0^2}{A' - B'}.$$

We investigate the system for Θ and φ (7) which has the first integral

$$\sin^2\Theta(\mu - \sin^2\varphi) = c_1 = \sin^2\Theta_0(\mu - \sin^2\varphi_0) = \text{const}. \tag{8}$$

The ranges of the variables Θ and φ in this system are $0 \le \Theta \le \pi$, and $0 \le \varphi < 2\pi$; the parameter μ can assume arbitrary values: $-\infty < \mu < +\infty$, depending on the relationships between the moments of inertia. We divide the domain D of μ, c_1 into three subdomains D_1, D_2 and D_3. The subdomain D_1 is specified by the inequalities $\mu \ge c_1 \ge 0$ $(\mu \ge 1)$; the subdomain D_2 is specified by the inequalities $\mu \ge c_1 \ge \mu - 1$ $(0 \le \mu \le 1)$; the subdomain D_3 is specified by the inequalities $0 \ge c_1 \ge \mu - 1$ $(\mu \le 0)$. The domain $D = D_1 \cup D_2 \cup D_3$ is shown in Figure 1.

The boundaries of the subdomains D_1, D_2, and D_3 are the singular sets for system (7). The motion corresponding to domains D_1 and D_3 is oscillatory in Θ and oscillatory or rotational in φ. The separatrix for the domain D_1 is given by $\sin^2\Theta = (\mu - 1) \times \times (\mu - \sin^2\varphi)^{-1}$, and for the domain D_3 it is given by $\sin^2\Theta = \mu(\mu - \sin^2\varphi)^{-1} \le 1$. In the domain D_2 oscillations occur both in Θ and φ.

There are 11 distinctive cases for the choice of the parameter μ [2]. Figure 2 shows the graphs of Θ versus φ obtained numerically on the basis of the first integral (7), for $\mu = -1.7$. According to these graphs, only oscillations occur in the variable Θ; in the variable φ, oscillations occur within the separatrix $\sin^2\Theta = \mu(\mu - \sin^2\varphi)^{-1}$ and rotations occur outside this separatrix.

Consider the special cases of the body motion. The value $\Theta = 0$ is the stationary point of the first equation (7). If $\Theta = 0$, then the differential equation for φ admits

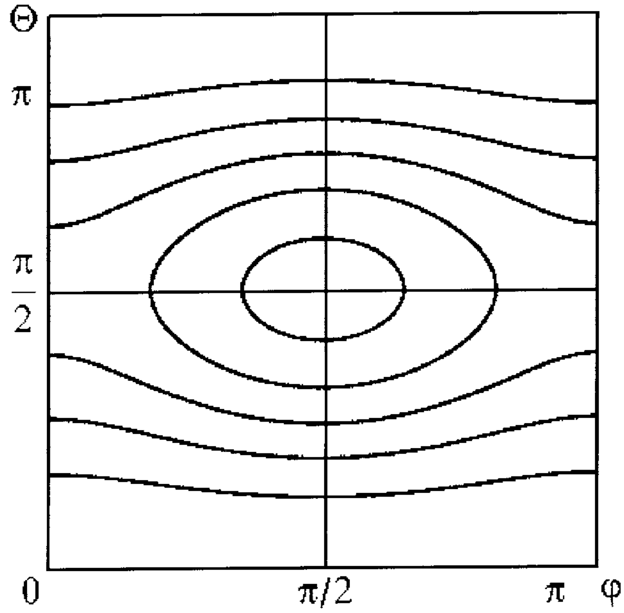

Figure 2. The graph of Θ *versus* φ for $\mu = 1.7$.

the separation of variables. On integrating we obtain the expression

$$\operatorname{tg}\varphi = l \operatorname{tg}(\pm r\tau + \operatorname{arctg}(l^{-1}\operatorname{tg}\varphi_0)),$$

$$l = \sqrt{\frac{\mu}{\mu - 1}}, \quad r = \sqrt{\mu(\mu - 1)}. \tag{9}$$

Upper and lower signs correspond to the cases $\mu > 1$ and $\mu < 0$. If $0 < \mu < 1$, then we have

$$\operatorname{tg}\varphi = j\frac{e^{J\tau}a - w}{e^{J\tau}a + w}, \tag{10}$$

$$j = \sqrt{\frac{\mu}{1 - \mu}}, \quad J = 2\sqrt{\mu(1 - \mu)},$$

$$a = j^{-1}\operatorname{tg}\varphi_0 + 1, \quad w = -(j^{-1}\operatorname{tg}\varphi_0 - 1).$$

For small Θ, the system (7) becomes: $\Theta' = \Theta \sin\varphi\cos\varphi$, $\varphi' = \mu - \sin^2\varphi$. In these equations, the terms of the order higher than linear in Θ are omitted. For small Θ, the equation for φ coincides with the corresponding equation for $\Theta = 0$, and its solution can be represented in the form (9), (10). On integrating the equation for Θ with allowance for (9), we obtain

$$\Theta^2 = \Theta_0^2 l^{\pm 2}(l^2\cos^2\varphi_0 + \sin^2\varphi_0)^{\mp 1}$$

$$\times \left[\cos^2(r\tau + \operatorname{arctg}(l^{-1}\operatorname{tg}\varphi_0)) + l^2\sin^2(r\tau + \operatorname{arctg}(l^{-1}\operatorname{tg}\varphi_0))\right]^{\mp 1}. \quad (11)$$

Upper and lower signs in (11) correspond to $\mu < 0$ and $\mu > 1$ respectively.

If $0 < \mu < 1$, we define with regard to (10)

$$\Theta^2 = \Theta_0^2 \cos^2\varphi_0 \frac{Ae^{2J\tau} + Be^{J\tau} + C}{4e^{J\tau}}, \quad (12)$$

where $A = (1 + j^2)a^2$, $B = 2aw(1 - j^2)$, $C = w^2(1 + j^2)$.

Thus, the evolution of rotations of a nearly dynamically spherical rigid body, containing a viscoelastic element, is investigated.

3. The Influence of Small Perturbing Moments with Definite Hieararchy of Smallness of Components

Consider the motion of a dynamically symmetric rigid body about a fixed point O under the action of the restoring moment depending on the angle of nutation Θ and the perturbing moment that is slowly changing in time. The equations of motion have the form [3]

$$\begin{aligned}
&A\dot{p} + (C - A)qr = k(\Theta)\sin\Theta\cos\varphi + M_1,\\
&A\dot{q} + (A - C)pr = -k(\Theta)\sin\Theta\sin\varphi + M_2,\\
&C\dot{r} = M_3,\\
&M_i = M_i(p, q, r, \psi, \Theta, \varphi, \tau), \quad \tau = \varepsilon t \quad (i = 1, 2, 3),\\
&\dot{\psi} = p\sin\varphi + q\cos\varphi,\\
&\dot{\Theta} = p\cos\varphi - q\sin\varphi,\\
&\dot{\varphi} = r - (p\sin\varphi + q\cos\varphi)\operatorname{ctg}\Theta.
\end{aligned} \quad (13)$$

Here p, q, r are the projections of the angular velocity vector on the principle axes of inertia; M_i $(i = 1, 2, 3)$ are the projections of the vector of the perturbing moment on the same axes, depending on the slow time $\tau = \varepsilon t$ ($\varepsilon \ll 1$ is the small parameter) and assumed to be the periodic functions of the Euler angles ψ, Θ, φ with periods 2π; and A is the equatorial and C is the axial moment of inertia about the point O, $A \neq C$.

We make the following assumptions:

$$p^2 + q^2 \ll r^2, \quad Cr^2 \gg k, \quad |M_i| \ll k \quad (i = 1, 2), \quad M_3 \sim k, \quad (14)$$

which mean that the direction of the angular velocity of the body is close to the axis of dynamic symmetry; the angular velocity is large; and two projections of the vector of the perturbing moment onto the principal axes of inertia of the body are small as compared to the restoring moment, while the third is of the same order as it. On the basis of inequalities (14), we introduce the small parameter ε and set

$$\begin{aligned} p &= \varepsilon P, \quad q = \varepsilon Q, \quad k(\Theta) = \varepsilon K(\Theta), \quad \varepsilon \ll 1, \\ M_i &= \varepsilon^2 M_i^*(P, Q, r, \psi, \Theta, \varphi, \tau), \quad (i = 1, 2), \\ M_3 &= \varepsilon^2 M_3^*(P, Q, r, \psi, \Theta, \varphi, \tau), \quad \tau = \varepsilon t. \end{aligned} \tag{15}$$

The problem that we pose is that of investigating the asymptotic behavior of the solutions of the system (13) for small ε, when conditions (14) and (15) are satisfied. To solve this problem, the averaging method [4] on the time interval of order ε^{-1} is used.

We make the transformation of variables

$$\begin{aligned} P &= a\cos\gamma + b\sin\gamma + KC^{-1}r^{-1}\sin\Theta\sin\varphi, \\ Q &= a\sin\gamma - b\cos\gamma + KC^{-1}r^{-1}\sin\Theta\cos\varphi, \\ a &= P_0 - K_0 C^{-1} r_0^{-1}\sin\Theta_0\sin\varphi_0, \\ b &= -Q_0 + K_0 C^{-1} r_0^{-1}\sin\Theta_0\cos\varphi_0, \quad K_0 = K(\Theta_0). \end{aligned} \tag{16}$$

Here $r_0, \psi_0, \Theta_0, \varphi_0, P_0$ and Q_0 are constants equal to the initial values of the variables at $t = 0$. The additional variable γ is defined by the equation

$$\dot{\gamma} = n, \quad \gamma(0) = 0, \quad n = (C - A)A^{-1}r. \tag{17}$$

Using formulas (16), we transform the variables $P, Q, r, \psi, \Theta, \varphi, \gamma$ in the system (13) to new variables $a, b, r, \psi, \Theta, \alpha, \gamma$, where

$$\alpha = \gamma + \varphi. \tag{18}$$

After performing the manipulations, we obtain the system of seven equations that is more convenient for what follows:

$$\begin{aligned} \dot{a} &= \varepsilon A^{-1}(M_1^0\cos\gamma + M_2^0\sin\gamma) + \varepsilon KC^{-2}r^{-2}M_3^0\sin\Theta\sin\alpha \\ &\quad - \varepsilon KC^{-1}r^{-1}\cos\Theta(b - KC^{-1}r^{-1}\sin\Theta\cos\alpha) \\ &\quad - \varepsilon C^{-1}r^{-1}\sin\Theta\sin\alpha(a\cos\alpha + b\sin\alpha)\frac{\mathrm{d}K}{\mathrm{d}\Theta}, \\ \dot{b} &= \varepsilon A^{-1}(M_1^0\sin\gamma - M_2^0\cos\gamma) - \varepsilon KC^{-2}r^{-2}M_3^0\sin\Theta\cos\alpha \\ &\quad + \varepsilon KC^{-1}r^{-1}\cos\Theta(a + KC^{-1}r^{-1}\sin\Theta\sin\alpha) \\ &\quad + \varepsilon C^{-1}r^{-1}\sin\Theta\cos\alpha(a\cos\alpha + b\sin\alpha)\frac{\mathrm{d}K}{\mathrm{d}\Theta}, \\ \dot{r} &= \varepsilon C^{-1}M_3^0, \quad \dot{\gamma} = (C - A)A^{-1}r, \quad \dot{\Theta} = \varepsilon(a\cos\alpha + b\sin\alpha), \\ \dot{\psi} &= \varepsilon(a\sin\alpha - b\cos\alpha)\operatorname{cosec}\Theta + \varepsilon KC^{-1}r^{-1}\cos\Theta, \\ \dot{\alpha} &= CA^{-1}r - \varepsilon(a\sin\alpha - b\cos\alpha)\operatorname{ctg}\Theta - \varepsilon KC^{-1}r^{-1}\cos\Theta. \end{aligned} \tag{19}$$

Here, M_i^0 denote functions obtained from M_i^* (see (15)) as a result of substitution (16), (18), i.e.,

$$M_i^0(a, b, r, \psi, \Theta, \alpha, \gamma, \tau) = M_i^*(P, Q, r, \psi, \Theta, \varphi, \tau), \quad (i = 1, 2, 3). \quad (20)$$

Let us consider the system (19) from the standpoint of employing the averaging method [4]. In [3, 5] the motions of a rigid body are investigated for assumption (14) when the perturbing moments do not depend on t and the restoring moment is $k = \text{const}$ or $k = k(\Theta)$. If to assume that the perturbing moments depend on fast variable t, then we obtain the essentially nonlinear system and the direct application of the averaging method is very difficult.

We investigate the case, when the perturbing moments depend on slow time $\tau = \varepsilon t, t \in [0, \varepsilon^{-1})$. The the system (19) contains the slow variables $a, b, r, \psi, \Theta, \tau$ and the fast variables, namely the phases α and γ. The functions M_i^0 $(i = 1, 2, 3)$ in (20) are 2π-periodic in α and γ. Then the system (19) contains two rotating phases α and γ, and the corresponding frequencies $CA^{-1}r$ and $(C - A)A^{-1}r$ are variable. At averaging the system (19) we should distinguish two cases: the nonresonant case, when the frequencies $CA^{-1}r$ and $(C - A)A^{-1}r$ are not commensurable (C/A is an irrational number), and the resonant case, when they are commensurable ($C/A = i/j$, $i/j \leq 2$, where i and j are natural relatively prime numbers). A very important feature of the system (19) is the fact that the frequency ratio is constant: $[(C - A)A^{-1}r]/[CA^{-1}r] = 1 - AC^{-1}$. The averaging of the nonlinear system (19) is equivalent to the averaging of a quasi-linear system with constant frequencies.

In the nonresonant case ($C/A \neq i/j$), we obtain the first-approximation averaged system by the independent averaging of the right-hand sides of the system (19) with respect to both fast variables. As a result, we obtain the following equations for the slow variables

$$\begin{aligned}
a' &= A\mu_1 - bKC^{-1}r^{-1}\cos\Theta + KC^{-2}r^{-2}\mu_3^s \sin\Theta \\
&\quad - \frac{1}{2}bC^{-1}r^{-1}\sin\Theta\frac{\mathrm{d}K}{\mathrm{d}\Theta}, \\
b' &= A^{-1}\mu_1 + aKC^{-1}r^{-1}\cos\Theta - KC^{-2}r^{-2}\mu_3^c \sin\Theta \\
&\quad + \frac{1}{2}aC^{-1}r^{-1}\sin\Theta\frac{\mathrm{d}K}{\mathrm{d}\Theta}, \\
r' &= C^{-1}\mu_3, \quad \psi' = KC^{-1}r^{-1}, \quad \Theta' = 0, \\
\mu_1 &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}(M_1^0\cos\gamma + M_2^0\sin\gamma)\,\mathrm{d}\alpha\,\mathrm{d}\gamma, \\
\mu_2 &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}(M_1^0\sin\gamma - M_2^0\cos\gamma)\,\mathrm{d}\alpha\,\mathrm{d}\gamma,
\end{aligned}$$

$$\mu_3 = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} M_3^0 \,\mathrm{d}\alpha\,\mathrm{d}\gamma,$$

$$\mu_3^s = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} M_3^0 \sin\alpha\,\mathrm{d}\alpha\,\mathrm{d}\gamma, \quad \mu_3^c = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} M_3^0 \cos\alpha\,\mathrm{d}\alpha\,\mathrm{d}\gamma. \tag{21}$$

Here $(\ldots)' = \mathrm{d}/\mathrm{d}\tau$, $\tau = \varepsilon t$.

We assume that the perturbing moments M_i $(i = 1, 2, 3)$ are linear-dissipative and write them with allowance for expressions (15) for p and q:

$$M_1 = -\varepsilon^2 I_1(\tau)P, \quad M_2 = -\varepsilon^2 I_1(\tau)Q, \quad M_3 = -\varepsilon I_3(\tau)r. \tag{22}$$

Here $I_1(\tau)$, $I_3(\tau)$ are positive integrable functions on the interval [0, 1).

After trasforming to new slow variables a, b, r, ψ, Θ we obtain the averaged system (21) of the form

$$\begin{aligned}
a' &= -I_1(\tau)A^{-1}a - C^{-1}r^{-1}b\left(K\cos\Theta + \frac{1}{2}\sin\Theta\frac{\mathrm{d}K}{\mathrm{d}\Theta}\right),\\
b' &= -I_1(\tau)A^{-1}b + C^{-1}r^{-1}a\left(K\cos\Theta + \frac{1}{2}\sin\Theta\frac{\mathrm{d}K}{\mathrm{d}\Theta}\right),\\
r' &= -I_3(\tau)C^{-1}r, \quad \psi' = KC^{-1}r^{-1}, \quad \Theta' = 0.
\end{aligned} \tag{23}$$

The solution of the first-approximation system for the slow variables in this case has a view

$$\begin{aligned}
a &= \exp\left(-A^{-1}\int_0^{\varepsilon t} I_1(\tau)\,\mathrm{d}\tau\right)(P_0\cos\beta + Q_0\sin\beta\\
&\quad - K_0C^{-1}r_0^{-1}\sin\Theta_0\sin(\beta+\varphi_0)),\\
b &= \exp\left(-A^{-1}\int_0^{\varepsilon t} I_1(\tau)\,\mathrm{d}\tau\right)(P_0\sin\beta - Q_0\cos\beta\\
&\quad + K_0C^{-1}r_0^{-1}\sin\Theta_0\cos(\beta+\varphi_0)),\\
\beta &= C^{-1}r_0^{-1}\left(K_0\cos\Theta_0 + \frac{1}{2}\sin\Theta_0\left.\frac{\mathrm{d}K}{\mathrm{d}\Theta}\right|_{\Theta=\Theta_0}\right)\\
&\quad\times\int_0^{\varepsilon t}\exp\left(C^{-1}\int_0^{\varepsilon t} I_3(\tau)\,\mathrm{d}\tau\right)\mathrm{d}\tau,
\end{aligned}$$

$$r = r_0 \exp\left(-C^{-1}\int_0^{\varepsilon t} I_3(\tau)\,\mathrm{d}\tau\right), \quad \Theta = \Theta_0, \quad K_0 = K(\Theta_0),$$

$$\psi = \psi_0 + KC^{-1}r_0^{-1}\int_0^{\varepsilon t} \exp\left(C^{-1}\int_0^{\varepsilon t} I_3(\tau)\,\mathrm{d}\tau\right)\mathrm{d}\tau. \tag{24}$$

As a result of substitution of the expressions for a, b and r from (24) into expressions (16) and (15) for P, Q, r, q we have

$$\begin{aligned} p = {} & \exp\left(-A^{-1}\int_0^{\varepsilon t} I_1(\tau)\,\mathrm{d}\tau\right)\big(p_0\cos(\gamma-\beta) - q_0\sin(\gamma-\beta) \\ & + k_0C^{-1}r_0^{-1}\sin\Theta_0\sin(\gamma-\beta-\varphi_0)\big) \\ & + kC^{-1}r_0^{-1}\sin\Theta_0\sin\varphi\exp\left(C^{-1}\int_0^{\varepsilon t} I_3(\tau)\,\mathrm{d}\tau\right), \\ q = {} & \exp\left(-A^{-1}\int_0^{\varepsilon t} I_1(\tau)\,\mathrm{d}\tau\right)\big(p_0\sin(\gamma-\beta) + q_0\cos(\gamma-\beta) \\ & - k_0C^{-1}r_0^{-1}\sin\Theta_0\cos(\gamma-\beta-\varphi_0)\big) \\ & + kC^{-1}r_0^{-1}\sin\Theta_0\cos\varphi\exp\left(C^{-1}\int_0^{\varepsilon t} I_3(\tau)\,\mathrm{d}\tau\right), \\ \gamma = {} & (C-A)A^{-1}r_0\int_0^{\tau}\exp\left(-C^{-1}\int_0^{\varepsilon t} I_3(\tau)\,\mathrm{d}\tau\right)\mathrm{d}t. \end{aligned} \tag{25}$$

We should note certain qualitative features of motion in this case. The modulus of the axial rotational velocity r decreases monotonically in the exponential fashion. The increment of the precession angle $\psi - \psi_0$ slowly increases exponentially. The slow variables a and b are the products of the exponentially decreasing factor and the oscillating factor. That all follows from (24).

In accordance with (25), the terms of the projections p and q, resulting from the initial values p_0 and q_0, attenuate exponentially and oscillate. At the same time, the projections p and q contain the exponentially increasing terms that are proportional to the restoring moment, thus leading to the exponential increase in $(p^2 + q^2)^{1/2}$.

4. The Influence of Small Perturbing Moments as Compared to the Restoring Ones

Consider the motion of a dynamically symmetric rigid body about a fixed point O due to a perturbing moment and a restoring moment (13).

We make the following assumptions

$$p^2 + q^2 \ll r^2, \quad Cr^2 \gg k, \quad |M_i| \ll k \quad (i = 1, 2, 3), \tag{26}$$

which mean that the direction of the angular velocity of the body is close to the axes of dynamic symmetry; the angular velocity is large enough; and the perturbing moments are small as compared to the restoring ones. Inequalities (26) justify the introduction of the small parameter $\varepsilon \ll 1$, so that

$$\begin{aligned} p &= \varepsilon P, \quad q = \varepsilon Q, \quad k(\Theta) = \varepsilon K(\Theta), \\ M_i &= \varepsilon^2 M_i^*(P, Q, r, \psi, \Theta, \varphi, \tau), \quad \tau = \varepsilon t \quad (i = 1, 2, 3). \end{aligned} \tag{27}$$

The problem of investigating the asymptotic behavior of the system (13) for small ε and conditions (27) is observed. We employ the averaging method [4] on the time interval of order ε^{-1}.

After a number of transformations of the system (13) we obtain the system of the form [6]

$$\begin{aligned} \dot{a} &= \varepsilon A^{-1}(M_1^0 \cos\gamma + M_2^0 \sin\gamma) \\ &\quad - \varepsilon K C^{-1} r_0^{-1} \cos\Theta (b - K C^{-1} r_0^{-1} \sin\Theta \cos\alpha) \\ &\quad - \varepsilon C^{-1} r_0^{-1} \sin\Theta \sin\alpha (a \cos\alpha + b \sin\alpha) \frac{\mathrm{d}K}{\mathrm{d}\Theta} \\ &\quad + \varepsilon^2 K C^{-1} r_0^{-2} \delta \cos\Theta (b - 2K C^{-1} r_0^{-1} \sin\Theta \cos\alpha) \\ &\quad + \varepsilon^2 C^{-1} r_0^{-2} \delta \sin\Theta \sin\alpha (a \cos\alpha + b \sin\alpha) \frac{\mathrm{d}K}{\mathrm{d}\Theta} \\ &\quad + \varepsilon^2 K C^{-2} r_0^{-2} M_3^0 \sin\Theta \sin\alpha, \\ \dot{b} &= \varepsilon A^{-1}(M_1^0 \sin\gamma - M_2^0 \cos\gamma) \\ &\quad + \varepsilon K C^{-1} r_0^{-1} \cos\Theta (a + K C^{-1} r_0^{-1} \sin\Theta \sin\alpha) \\ &\quad + \varepsilon C^{-1} r_0^{-1} \sin\Theta \cos\alpha (a \cos\alpha + b \sin\alpha) \frac{\mathrm{d}K}{\mathrm{d}\Theta} \\ &\quad - \varepsilon^2 K C^{-1} r_0^{-2} \delta \cos\Theta (a + 2K C^{-1} r_0^{-1} \sin\Theta \sin\alpha) \\ &\quad - \varepsilon^2 C^{-1} r_0^{-2} \delta \sin\Theta \cos\alpha (a \cos\alpha + b \sin\alpha) \frac{\mathrm{d}K}{\mathrm{d}\Theta} \\ &\quad - \varepsilon^2 K C^{-2} r_0^{-2} M_3^0 \sin\Theta \cos\alpha, \end{aligned}$$

$$\begin{aligned}
\dot{\delta} &= \varepsilon C^{-1} M_3^0, \quad M_i^0(a, b, \delta, \psi, \Theta, \alpha, \gamma, \tau) = M_i^*(P, Q, \delta, \psi, \Theta, \varphi, \tau), \\
\dot{\psi} &= \varepsilon(a \sin\alpha - b\cos\alpha)\operatorname{cosec}\Theta + \varepsilon K C^{-1} r_0^{-1} - \varepsilon^2 K C^{-1} r_0^{-2}\delta, \\
\dot{\gamma} &= (C - A)A^{-1} r_0 + \varepsilon(C - A)A^{-1}\delta, \quad \dot{\Theta} = \varepsilon(a\cos\alpha + b\sin\alpha), \\
\dot{\alpha} &= CA^{-1} r_0 + \varepsilon C A^{-1}\delta - \varepsilon(a\sin\alpha - b\cos\alpha)\operatorname{ctg}\Theta \\
&\quad - \varepsilon K C^{-1} r_0^{-1}\cos\Theta + \varepsilon^2 K C^{-1} r_0^{-2}\delta\cos\Theta. \qquad (28)
\end{aligned}$$

In the system (28), $a, b, \delta, \psi, \Theta$ are the slow variables and α, γ are the fast variables.

In [6, 7], the motions of the rigid body are investigated at assumption (26), that the perturbing moments do not depend on t and the restoring moment is $k = \text{const.}$ or $k = k(\Theta)$.

We investigate the case of dependence of perturbing moments on slow time $\tau = \varepsilon t$, $t \in [0, \varepsilon^{-1})$. Functions M_i^0 $(i = 1, 2, 3)$ in (28) are periodic in α and γ with the period 2π.

In a number of studies, for example [8, 9], perturbing motions of the rigid body, similar to Lagrange case under the action of the moment slowly changing in time, are investigated.

Let us consider the perturbed Lagrange motion, allowing for the moments applied to the body from the external medium. We assume that the perturbing moments M_i $(i = 1, 2, 3)$ (see (27)) have the form

$$M_1 = -\varepsilon^2 I_1(\tau)P, \quad M_2 = -\varepsilon^2 I_1(\tau)Q, \quad M_3 = -\varepsilon^2 I_3(\tau)r, \qquad (29)$$

where $I_1(\tau)$, $I_3(\tau)$ are the positive integrable functions on the interval [0, 1).

After several transformations the solution of the averaged system of first-approximation equations for the slow and the fast variables in the case (29) has the form

$$\begin{aligned}
a^{(1)} &= \exp\left(-A^{-1}\int_0^{\varepsilon t} I_1(\tau)\,\mathrm{d}\tau\right)(a^0\cos wt - b^0\sin wt), \\
b^{(1)} &= \exp\left(-A^{-1}\int_0^{\varepsilon t} I_1(\tau)\,\mathrm{d}\tau\right)(b^0\cos wt + a^0\sin wt), \\
\delta^{(1)} &= -C^{-1} r_0 \int_0^{\varepsilon t} I_3(\tau)\,\mathrm{d}\tau, \\
\psi^{(1)} &= \varepsilon K C^{-1} r_0^{-1} t + \psi_0, \quad \Theta^{(1)} = \Theta_0, \\
\alpha^{(1)} &= CA^{-1} r_0 t - \varepsilon A^{-1}\int_0^t \left(\int_0^{\varepsilon t} I_3(\tau)\,\mathrm{d}\tau\right)\mathrm{d}t
\end{aligned}$$

$$- \varepsilon K C^{-1} r_0^{-1} \cos\Theta_0 t + \varphi_0,$$

$$\gamma^{(1)} = (C-A)A^{-1} r_0 t - \varepsilon (C-A) A^{-1} C^{-1} r_0 \int_0^t \left(\int_0^{\varepsilon t} I_3(\tau)\,\mathrm{d}\tau \right) \mathrm{d}t. \quad (30)$$

Here

$$w = \frac{1}{2} \varepsilon C^{-1} r_0^{-1} \left(2K \cos\Theta_0 + \sin\Theta_0 \frac{\mathrm{d}K}{\mathrm{d}\Theta} \right);$$

the quantities a_0, b_0 are determined by the following way: $a = P_0 - \lambda_0 \sin\varphi_0$, $b = -Q_0 + \lambda_0 \cos\varphi_0$, $\lambda_0 = K_0 C^{-1} r_0^{-1} \sin\Theta_0$; the variable $\gamma = \gamma_0$ has the meaning of the phase of the oscillations, $\alpha = \gamma + \varphi$, $r = r_0 + \varepsilon\delta$; P_0, Q_0, r_0, ψ_0, φ_0 are the initial values of the appropriate variables at $t = 0$.

We can determine the evolution of the precession and the nutation angles in the second approximation:

$$\Theta_\varepsilon^\nu(t) = \Theta_0 + \varepsilon A C^{-1} r_0^{-1} \exp\left(-A^{-1} \int_0^{\varepsilon t} I_1(\tau)\,\mathrm{d}\tau \right) C^0 \sin(\alpha^{(1)} - \chi)$$

$$+ \varepsilon K_0 C^{-2} r_0^{-2} \int_0^{\varepsilon t} I_1(\tau)\,\mathrm{d}\tau \sin\Theta_0,$$

$$\psi_\varepsilon^\nu(t) = \psi_0 + \varepsilon K C^{-1} r_0^{-1} t + S^{(1)},$$

$$S^{(1)} = \varepsilon^2 K C^{-2} r_0^{-1} \int_0^t \left(\int_0^{\varepsilon t} I_3(\tau)\,\mathrm{d}\tau \right) \mathrm{d}t + \varepsilon^2 A K^2 C^{-3} r_0^{-3} \cos\Theta_0 t$$

$$- \varepsilon A C^{-1} r_0^{-1} \exp\left(-A^{-1} \int_0^{\varepsilon t} I_1(\tau)\,\mathrm{d}\tau \right) C^0 \sin(\alpha^{(1)} + \beta) \operatorname{cosec}\Theta_0,$$

$$\sin\chi = \cos\beta = b^{(1)} (C^0)^{-1} \exp\left(A^{-1} \int_0^{\varepsilon t} I_1(\tau)\,\mathrm{d}\tau \right),$$

$$C^0 = \sqrt{(a^0)^2 + (b^0)^2}. \quad (31)$$

Thus the term of order ε in the expression (31) for Θ_ε^ν is the product of the exponentially decreasing factor

$$\exp\left(-A^{-1} \int_0^{\varepsilon t} I_1(\tau)\,\mathrm{d}\tau \right),$$

caused by the energy dissipation, and the oscillating factor $\sin(\alpha^{(1)} - \chi)$. The magnitude of the attenuation decrement and the behaviour of the slow phase of small oscillations are evident from formulas (30) for $a^{(1)}$, $b^{(1)}$. The term $S^{(1)}$ of order $O(\varepsilon)$ in the expression (31) for $\psi_\varepsilon^\nu(t)$ gives a more precise definition to the formula for the angular precession velocity $\omega_p = KC^{-1}r_0^{-1}$, obtained in the approximate gyroscope theory [10].

References

1. Chernousko, F.L., 'Motion of the rigid body with moving internal masses', *Mechanics of Rigid Body* **8**(4), 1973, 33–44.
2. Akulenko, L.D. and Leshchenko, D.D., 'Evolution of rotation of a nearly dynamically spherical triaxial satellite under the action of light pressure torques', *Mechanics of Rigid Body* **31**(2), 1996, 3–12.
3. Leshchenko, D.D. and Sallam, S.N., 'Perturbed rotational motions of a rigid body relative to a fixed point', *Mechanics of Rigid Body* **25**(5), 1990, 16–23.
4. Volosov, V.M. and Morgunov, B.I., *Method of Averaging in the Theory of Nonlinear Oscillatory Systems*, Publishing House of Moscow State University, Moscow, 1971 [in Russian].
5. Akulenko, L.D., Leshchenko, D.D. and Chernousko, F.L. 'Perturbed motion of a rigid body, close to regular precession', *Mechanics of Rigid Body* **21**(5), 1986, 3–10.
6. Leshchenko, D.D. and Sallam, S.N., 'Perturbed rotational motions of a rigid body, similar to regular precession', *Applied Mathematics and Mechanics (PMM)* **54**(2), 1990, 224–232.
7. Leshchenko, D.D. and Shamaev, A.S., 'Perturbed rotational motions of a rigid body that are close to regular precession in the Lagrange case', *Mechanics of Rigid Body* **22**(6), 1987, 8–17.
8. Kuzmak, G.E., 'To the question about the space motion of the axially symmetric rigid body about a fixed point under the action of moments that are slowly changing in time', *Doklady of the USSR Academy of Sciences* **132**, 1960, 549–552.
9. Moiseev, N.N., *Asymptotic Methods of Nonlinear Mechanics*, Nauka, Moscow, 1981 [in Russian].
10. Bukhgolts, N.N., *An Elementary Course of Theoretical Mechanics*, Vol. 2, Nauka, Moscow, 1969 [in Russian].