Perturbed rotational motions of a rigid body

Article · January 1990

Dmytro Leshchenko
Odessa State Academy of Civil Engineering and Architecture

THE LYAPUNOV functions method and applications
P. Borne and V. Matrosov (editors)
J.C. Baltzer AG, Scientific Publishing Co. © IMACS, 1990
pp. 227-232

PERTURBED ROTATIONAL MOTIONS OF A RIGID BODY

D.D. LESHCHENKO
Institute of Low Temperature Technology and Energetics, 270057 Odessa, Petra Velikogo 1/3, USSR

1. INTRODUCTION

The author investigate perturbed rotational motions of a rigid body that are close to regular precession in the Lagrange case when the restoring moment depends on the nutation angle. It is assumed that the angular velocity of the body is large, its direction is close to the axis of dynamic symmetry of the body, and that two projections of the vector of the perturbing moment onto the principal axes of inertia of the body are small as compared to the restoring moment, while the third is of the same order of magnitude as the moment in question. A small parameter is introduced in a special way; the averaging method is employed. The averaged system of equations of motion is obtained in first approximation. Examples are considered.

2. MATERIALS AND METHODS

Consider the motion of dynamically symmetrical rigid body about fixed point O under the action of restoring moment depending on the nutation angle θ and perturbing moment. The equations of motion have the form

$$Ap^{\cdot} + (C - A)qr = k(\theta) \sin\theta \cos\varphi + M_1$$

$$Aq^{\cdot} + (A - C)pr = -k(\theta) \sin\theta \sin\varphi + M_2 \qquad (1)$$

$$Cr^{\cdot} = M_3, \quad M_i = M_i(p,q,r,\psi,\theta,\varphi,t) \quad (i = 1,2,3)$$

$$\psi^{\cdot} = (p \sin\varphi + q \cos\varphi) \operatorname{cosec}\theta,$$

$$\theta^{\cdot} = p \cos\varphi - q \sin\varphi,$$

$$\varphi^{\cdot} = r - (p \sin\varphi + q \cos\varphi) \operatorname{ctg}\theta$$

Dynamic equations (1) are written in projections onto the principal axes of inertia of the body, passing through point O. Here p, q, r are the projections of the angular velocity vector of the body onto these axes, M_i $(i = 1,2,3)$ are the projections of the vector of the perturbing moment onto these same axes, which are 2π-periodic functions of the Euler angles ψ, θ, φ; and A and C are the equatorial and axial moments of inertia of the body relative to point O, $A \neq C$.

The perturbing moments M_i in (1) are assumed to be known functions of their arguments. For $M_i = 0$ $(i = 1,2,3)$ and $k(\theta) = \text{const} = mgl$ equations (1) correspond to the Lanrange case. Here m is the mass of the body; g is acceleration due to gravity; and l is the distance from fixed point O to the center of gravity of the body.

We make the following initial assumptions:

$$p^2 + q^2 \ll r^2, \qquad Cr^2 \gg k \qquad (i = 1, 2),$$

$$M_i \ll k \qquad (i = 1, 2), \qquad M_3 \sim k \qquad (2)$$

which mean that the direction of the angular velocity of the body is close to the axis of dynamic symmetry; the angular velocity is large; two projections of the vector of the perturbing moment onto the principal axis of inertia of the body are small as compared to the restoring moment, while the third is of the same order of magnitude as this moment. Inequalities (2) allow us to introduce the small parameter ε and to set

$$p = \varepsilon P, \quad q = \varepsilon Q, \quad k(\theta) = \varepsilon K(\theta), \quad \varepsilon \ll 1$$

$$M_i = \varepsilon^2 M_i^*(P,Q,r,\psi,\theta,\varphi,t) \qquad (i = 1, 2) \qquad (3)$$

$$M_3 = \varepsilon M_3^*(P,Q,r,\psi,\theta,\varphi,t)$$

The problem that we formulate is that of investigating the asymptotic behavior of the solutions of system (1) for small ε, if conditions (2) and (3) are satisfied. This will be done by employing the averaging method [1, 2]. A number of studies, e.g. [3 - 5], have investigated perturbed motions close to Lagrange motion.

In system (1) we make change of variables (3). Let us consider the zero-approximation system; we set $\varepsilon = 0$. Then the last four obtained equations yield

$$r = r_0, \quad \psi = \psi_0, \quad \varphi = r_0 t + \varphi_0 \tag{4}$$

Here $r_0, \psi_0, \theta_0, \varphi_0$ are constants equal to the initial values of the corresponding variables for $t = 0$. We substitute (4) into the first two equations of system (1) with allowance for expressions (3) for $\varepsilon = 0$, and we integrate the resultant system of the equations for P, Q. We write the solution in the form

$$P = a \cos\gamma_0 + b \sin\gamma_0 + K_0 C^{-1} r_0^{-1} \sin\theta_0 \sin(r_0 t + \varphi_0)$$

$$Q = a \sin\gamma_0 - b \cos\gamma_0 + K_0 C^{-1} r_0^{-1} \sin\theta_0 \cos(r_0 t + \varphi_0)$$

$$a = P_0 - K_0 C^{-1} r_0^{-1} \sin\theta_0 \sin\varphi_0, \tag{5}$$

$$b = -Q_0 + K_0 C^{-1} r_0^{-1} \sin\theta_0 \cos\varphi_0,$$

$$\gamma_0 = n_0 t, \quad n_0 = (C - A)A^{-1} r_0 \neq 0,$$

$$|n_0/r_0| \leq 1, \quad K_0 = K(\theta_0)$$

Here P_0, Q_0 are the initial values of the new variables P, Q, introduced in accordance with (3). System (1) with allowance for expressions (3) is essentially nonlinear and therefore we introduce the additional variable γ, defined by the equation

$$\gamma^{\cdot} = n, \quad \gamma(0) = 0, \quad n = (c - A)^{-1} r\, A^{-I} \tag{6}$$

By eliminating the constants, with allowance for (4), it is possible to rewrite the first two expressions in (5) in equivalent form:

$$P = a \cos\gamma + b \sin\gamma + KC^{-1} r^{-1} \sin\theta \sin\varphi \tag{7}$$

$$Q = a \sin\gamma - b \cos\gamma + KC^{-1} r^{-1} \sin\theta \cos\varphi$$

and to solve for a, b:

$$a = P \cos\gamma + Q \sin\gamma - KC^{-1} r^{-1} \sin\theta \sin(\gamma + \varphi)$$

$$b = P \sin\gamma - Q \cos\gamma + KC^{-1} r^{-1} \sin\theta \cos(\gamma + \varphi)$$

Let us consider system (1) with allowance for expressions (3) for $\varepsilon \neq 0$. Using formulas (7), (8) in system (1) with allowance for expression (3), (6) we convert from the variables $P, Q, r, \psi, \theta, \varphi, \gamma$ to the new variables $a, b, r, \psi, \theta, \alpha, \gamma$, where

$$\alpha = \gamma + \varphi \tag{9}$$

After some manipulation, we obtain a system of seven equations

$$a^{\cdot} = \varepsilon A^{-1}(M_1^0 \cos\gamma + M_2^0 \sin\gamma) - \varepsilon KC^{-1} r^{-1} \cos\theta (b -$$
$$- KC^{-1} r^{-1} \sin\theta \cos\alpha) + \varepsilon KC^{-2} r^{-2} M_3^0 \sin\theta \sin\alpha -$$
$$- \varepsilon C^{-1} r^{-1} \sin\theta \sin\alpha (a \cos\alpha + b \sin\alpha)\, dK/d\theta$$

$$b^{\cdot} = \varepsilon A^{-1}(M_1^0 \sin\gamma - M_2^0 \cos\gamma) + \varepsilon KC^{-1} r^{-1} \cos\theta (a +$$
$$+ KC^{-1} r^{-1} \sin\theta \sin\alpha) - \varepsilon Kc^{-2} r^{-2} M_3^0 \sin\theta \cos\alpha +$$
$$+ \varepsilon C^{-1} r^{-1} \sin\theta \cos\alpha (a \cos\alpha + b \sin\alpha)\, dK/d\theta$$

$$r^{\cdot} = \varepsilon C^{-1} M_3^0, \tag{10}$$

$$\psi^{\cdot} = \varepsilon \operatorname{cosec}\theta (a \sin\alpha - b \cos\alpha) + \varepsilon KC^{-1} r^{-1}$$

$$\theta^{\cdot} = \varepsilon (a \cos\alpha + b \sin\alpha),$$

$$\alpha^{\cdot} = CA^{-1}r - \varepsilon \operatorname{ctg}\theta\,(a \sin\alpha - b\cos\alpha) -$$
$$-\varepsilon KC^{-1}r^{-1}\cos\theta, \qquad \gamma^{\cdot} = (c - A)A^{-1}r$$

Here M_i^0 denotes functions obtained from M_i^* (see (3)) as a result of substitution (7) - (9), i.e.,

$$M_i^0(a,b,r,\psi,\theta,\alpha,\gamma,t) = M_i^*(P,Q,r,\psi,\theta,\varphi,t) \qquad (i = 1, 2, 3) \tag{11}$$

System (10) contains the slow variables a,b,r,ψ,θ and fast variables represented by the phases α, γ and time t. Let us assume, for the sake of simplicity, that the perturbing moments M_i^* are independent of t. Since M_i^* (i = 1,2,3) are 2π-periodic in φ, it follows, in accordance with (7) - (9), that functions M_i^0 from (11) will be 2π-periodic functions of α and γ. Then system (10) contains two rotating phases α and γ and the corresponding frerquencies $CA^{-1}r$ and $(C - A)A^{-1}r$ are variable. In averaging system (10) two cases should be distinguished: the nonresonant case, when frequencises $CA^{-1}r$ and $(C-A)A^{-1}r$ are noncomensurable, and the resonant case, when these frequencies are comensurable [6]. A very important feature of system (10) is the fact that the ratio of the frequencies is constatnt $[(c-A)A^{-1}r]/[CA^{-1}r] = 1 - AC^{-1}$ and the resonant case occurs for

$$C/A = i/j, \qquad i/j \le 2 \tag{12}$$

where i and j are relatively prime natural numbers, while in the nonresonant case C/A is an irrational number. As a result of (12), averaging of nonlinear system (10), in which M_i^0 is independent of t, is equivalent to averaging of a quasilinear system with constant frequencies. This can be achieved by introducing the independent variable γ.

In the nonresonant case ($C/A \neq i/1$) we obtain the first-approximation averaged system by independent averaging of the right sides of system (10) with respect to both fast variables α, γ. As a result, we obtain the following equations for the slow variables:

$$a^{\cdot} = \varepsilon A^{-1}\mu_1 - \varepsilon bKC^{-1}r^{-1}\cos\theta + \varepsilon KC^{-2}r^{-2}\sin\theta\,\mu_3^S -$$
$$-1/2\,\varepsilon C^{-1}r^{-1}b\sin\theta\,dK/d\theta \tag{13}$$

$$b^{\cdot} = \varepsilon A^{-1}\mu_2 + \varepsilon aKC^{-1}r^{-1}\cos\theta - \varepsilon KC^{-2}r^{-2}\sin\theta\mu_3^c +$$
$$+ 1/2\,\varepsilon C^{-1}r^{-1}a\sin\theta\,dK/d\theta$$

$$r^{\cdot} = \varepsilon C^{-1}\mu_3, \quad \psi^{\cdot} = \varepsilon KC^{-1}r^{-1}, \qquad \theta^{\cdot} = 0$$

$$\mu_1 = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}(M_1^0\cos\gamma + M_2^0\sin\gamma)\,d\alpha\,d\gamma$$

$$\mu_2 = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}(M_1^0\sin\gamma - M_2^0\cos\gamma)\,d\alpha\,d\gamma$$

$$\mu_3 = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} M_3^0\,d\alpha\,d\gamma,$$

$$\mu_3^S = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} M_3^0\sin\alpha\,d\alpha\,d\gamma,$$

$$\mu_3^c = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} M_3^0\cos\alpha\,d\alpha\,d\gamma$$

Note that the last equation in system (13) can be integrated, it yields $\theta = \theta_0$.

As an example of restoring moment depending on the nutation angle consider rigid body with a spring connected to the point of the body N. The tip L of the spring is attached fixedly. The body is under the action of gravity force mg and elastic force of spring F, the modulus of which is proportional to the deformation of spring $F = \lambda(S - S_0)$. Here λ is stiffness coefficient of spring. In this case restoring moment have the form

$$k(\theta) = mgl + \lambda hz\,[1 - s_0(h^2 + z^2 - 2hz\cos\theta)^{-1/2}]$$

here ON = z, OC = l, OL = h, LN = S = S(θ). According to (3) $k(\theta) = \varepsilon K(\theta)$.

Let us consider perturbed Lagrange motion with allowance for the moments acting on our rigid body from the environment. We will assume that the perturbing moments M_i (i = 1,2,3) with allowance for expressions (3) for p and q have the form [7]

$$M_1 = -\varepsilon^2 I_1 P, \quad M_2 = -\varepsilon^2 I_1 Q, \qquad (15)$$

$$M_3 = -\varepsilon I_3 r, \qquad I_1,\ I_3 > 0$$

Here I_1, I_2 are constant proportionally factors that depend on the properties of the medium and the shape of the body.

For the nonresonant case we obtain averaged system (13) of the form

$$a^{\cdot} = -\varepsilon I_1 A^{-1} a - \varepsilon C^{-1} r^{-1} b(K\cos\theta + 1/2\sin\theta\, dK/d\theta)$$

$$(16)$$

$$b^{\cdot} = -\varepsilon I_1 A^{-1} b + \varepsilon C^{-1} r^{-1} a(K\cos\theta + 1/2\sin\theta\, dK/d\theta)$$

$$r^{\cdot} = -\varepsilon I_3 C^{-1} r, \qquad \psi^{\cdot} = \varepsilon K C^{-1} r^{-1}, \qquad \theta^{\cdot} = 0$$

Integrating the third equation in (16), we obtain

$$r = r_0 \exp(-\varepsilon I_3 C^{-1} t), \qquad r_0 \neq 0 \qquad (17)$$

Equation (16) for $\psi^{\cdot}$ can be integrated with allowance for (17); it yields

$$\psi = \psi_0 + K I_3^{-1} r_0^{-1}[\exp(\varepsilon I_3 C^{-1} t) - 1] \qquad (18)$$

here r_0, ψ_0 were obtained in (4). As can be seen from (16), the angle of nutation maintains constant value $\theta = \theta_0$.

Substituting (17) for r in the first two equations in (16), we obtain a system whose solution is described as follows:

$$a = \exp(-\varepsilon I_1 A^{-1} t)[P_0\cos\eta + Q_0\sin\eta -$$

$$- K_0 C^{-1} r_0^{-1}\sin\theta_0\sin(\eta + \varphi_0)] \qquad (19)$$

$$b = \exp(-\varepsilon I_1 A^{-1} t)[P_0\sin\eta - Q_0\cos\eta +$$

$$+ K_0 C^{-1} r_0^{-1}\sin\theta_0\cos(\eta + \varphi_0)]$$

$$\eta = r_o^{-1} I_3^{-1}(K\cos\theta +$$

$$+ 1/2\sin\theta\, dK/d\theta)[\exp(\varepsilon I_3 C^{-1} t) - 1]$$

$$K_0 = K(\theta_0)$$

As a result of substitution into expressions (7), (3) for P, Q, p, q of the expressions for a and b from (19) and for r from (17), we obtain

$$p = \exp(-\varepsilon I_1 A^{-1} t)[p_0 \cos(\gamma-\eta) - q_0 \sin(\gamma-\eta) +$$

$$+ k_0 C^{-1} r_0^{-1} \sin\theta_0 \sin(\gamma-\eta-\varphi_0)] +$$

$$+ kC^{-1} r_0^{-1} \exp(\varepsilon I_3 C^{-1} t) \sin\theta_0 \sin\varphi \qquad (20)$$

$$q = \exp(-\varepsilon I_1 A^{-1} t)[p_0 \sin(\gamma-\eta) +$$

$$+ q_0 \cos(\gamma-\eta) - k_0 C^{-1} r_0^{-1} \sin\theta_0 \cos(\gamma-\eta-\varphi_0) +$$

$$+ kC^{-1} r_0^{-1} \exp(\varepsilon I_3 C^{-1} t) \sin\theta_0 \cos\varphi$$

$$\gamma = \frac{C}{I_3} \frac{C-A}{A} \frac{r_0}{\varepsilon} [1 - \exp(-\varepsilon I_3 C^{-1} t)],\ p_0 = \varepsilon P_0,$$

$$q_0 = \varepsilon Q_0, \qquad k_0 = \varepsilon K_0$$

For the body with spring the expression (14) for the restoring moment $k = k(\theta)$ must be substitute into expressions (20) and this expression (14) for $k = k(\theta_0)$ when $\theta = \theta_0$. In this case

$$\eta = r_0^{-1} I_3^{-1} \varepsilon^{-1} [k \cos\theta + 1/2\lambda h^2 z^2 \sin^2\theta (h^2 + z^2 -$$

$$- 2hz \cos\theta)^{-3/2}] [\exp(\varepsilon I_3 C^{-1} t) - 1]$$

Let us point out some qualitative features of motion in the case in question. The modulus of the axial rotational velocity r decreases monotonically in exponential fashion in accordance with (17). The increment of the precession angle $\psi - \psi_0$ increases slowly exponentially in accordance with (18). It follows from (19) that the slow variables a and b tend monotonically to zero exponentially.

In accordance with (20), the terms of the projections p and q that are due to the initial values p_0, q_0, attenuate exponentially. At the same time, projections p and q contain exponentially increasing terms that are proportional to the restoring moment, with the result that the quantity $(p^2 + q^2)^{1/2}$ grows exponentially.

3. CONCLUSIONS

Perturbed rotational motions of a rigid body that are close to regular precession in the Lagrange case when the restoring moment depends on the nutation angle are investigated. It is assumed that the angular velocity of the body is large, its direction is close to the axis of dynamic symmetry of the body, and that two projections of the vector of the perturbing moment into the principal axes of inertia of the body are small as compared to the restoring moment, while the third is of the same order of magnitude as this moment. These assumptions allow us to introduce the small parameter, the averaging method is employed. The averaged system of equations of motion is obtained in first approximation in the nonresonant case. As an example of restoring moment depending on the nutation angle consider rigid body with a spring connected to the point of the body. Perturbed Lagrange motion with allowance for the linear-dissipative moments acting on rigid body from the environment is considered.

REFERENCES

[1] Bogolyobov, N.N. and Mitropol'skii, Yu.A., Asymptotic Methods in the Theory of Nonlinear Oscillations (Nauka, Moscow, 1974).

[2] Volosov V.M. and Morgunov B.I., Method of Averaging in the Theory of Nonlinear Oscillatory Systems (Izd-vo MGU, Moscow, 1971).

[3] Akulenko, L.D., Leshchenko, D.D. and Chernousko, F.L., PMM, Vol. 43, no 5 (1979) 771.

[4] Akulenko, L.D., Leshchenko, D.D., and Chernousko, F.L., Izv. AN SSSR. MTT (Mechanics of Solids), No. 5 (1986) 3.

[4] Akulenko, L.D., Leshchenko, D.D., and Chernousko, F.L., Izv. AN SSSR. MTT (Mechanics of Solids), No. 5 (1986) 3.
[5] Leshchenko, D.D., Shamaev, A.S., Izv. AN SSSR MTT (Mechanics of Solids), No. 6 (1987) 8.
[6] Arnol'd V.I., Supplementary Chapters in the Theory of Ordiniry Differential Equations (Nauka, Moscow, 1978).
[7] Koshlyakov V.N., Problems of Dynamics of Rigid Rodies and Applied Gyroscope Theory. Analytical Methods (Nauka, Moscow, 1985).