

# DEFLECTION AND BENDING MOMENTS AMPLITUDE DISTRIBUTION AT THE FORCED OSCILLATIONS OF THE EULER-BERNOULLI BEAM

Yurii KRUTII, Nikolay SURYANINOV

**Abstract:** The paper considers forced oscillations of a simply supported Euler-Bernoulli beam with inner inelastic resistance. Maximum values of non-dimensional amplitudes of bending moments and deflections which are invariants with respect to dimensional parameters of beam are calculated. For beams with any dimensional parameters calculation of the maximum amplitudes corresponding to the set frequency of forced vibrations, reduces to multiplication by appropriate dimensional factor already calculated invariant dimensionless values. In a specific example, a comparative analysis of accurate amplitude values for dynamic bending moments and deflections is calculated according to the author's method, with the same calculation in the ANSYS program complex. Values of the amplitudes in the vicinity of the resonance frequencies are clarified.

**Keywords:** amplitude of the bending moments and deflections, Euler-Bernoulli beam, transverse vibrations, frequency of forced oscillations, resonance.

## 1 INTRODUCTION

One of the major problems in the theory of oscillations of elastic systems is the study of the influence of external and internal resistances to oscillation processes. The calculations on the free oscillations consist in definition of own frequencies and forms of oscillations, as well as calculations on forced vibrations away from the resonance zones. The resistance forces are often neglected. This is due to the fact that when such calculations accounting the forces of resistance do not significantly influence the final result. Another thing is calculation of oscillation near resonance. This calculation requires the consideration of the resistance forces, since their effect is appearing in the greatest measure.

Effect of external and internal resistance to oscillations is different and depends on many factors: the oscillatory characteristics of the system, the material of which the elements of the system are made, parameters of environment. However, in this case, internal resistance of inelastic material is of particular importance.

## 2 BACKGROUND AND ANALYSIS OF RESEARCH

Presence of internal friction in the material was firstly found by Coulomb in experiments with a torsion balance. The study of internal friction involved W. Thomson. It was established experimentally that the resilient material does not strictly follow Hooke's law even if the elastic deformations within the limits of elasticity. This explains the internal energy loss fluctuations.

Many scientists talk about the importance of internal friction in the material studies conducted in the beginning of the 20<sup>th</sup> century. Experiments of Guye on the internal losses in the material during torsional vibration of metal wires showed a completely insignificant role of air friction as compared with losses in the metal. Rowett, exploring damping vibrations in machines, found that the share of

domestic energy dissipation in the material accounts for at least two-thirds of all losses during vibration.

In the dynamic structural analysis, the hypothesis of Kelvin-Voigt, which is based on the idea of the viscosity of solids, is widespread when taking into account the internal resistance of inelastic material [1, 2]. The internal friction proportional to the velocity was used by the founders of the applied theory of vibrations A.N. Krylov [3] and S.P. Timoshenko [4].

However, as it is well known [2], the Kelvin-Voigt's hypothesis in its pure form has a number of drawbacks. The main point is the fact that the hypothesis leads to a conclusion about the frequency-independent internal friction in the material, which contradicts the experimental data. This disadvantage is eliminated by taking the corrected Kelvin-Voigt hypothesis, according to which the damping is taken into account in proportion to the strain rate, divided by the oscillation frequency [2].

Among the fundamental studies on this issue, particularly noteworthy are the works of A.N. Krylov [3], S.P. Tymoshenko [4], J.G. Panovko [5] and E.S. Sorokin [6].

Current studies are characterized by the extensive use of computer methods of mechanics. Works related to considering variable mass for different kinds of friction are published by V.P. Olshansky and S.V. Olshansky [7, 8]. The work of N.N. Berendeev is devoted to the problem of the influence of the internal friction on the system of forced oscillations [9].

## 3 MAIN CHAPTER

### 3.1 The main symbols and formulas

Consider the forced harmonic oscillations of a hinged beam internal forces taking into account the inelastic resistance. The general scheme of the oscillation is shown in

Fig. 1. Fig. 2 shows a diagram of forces acting on the oscillations of the beam element.

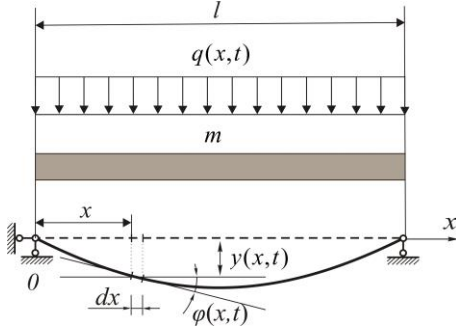


Figure 1 Forced transverse vibrations of a rod

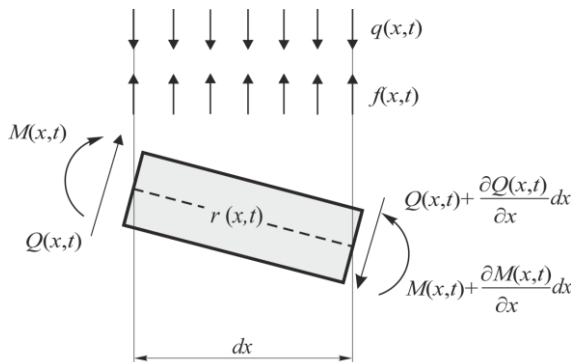


Figure 2 Driving forces acting on the rod element

The following designations are accepted:  $q(x,t) = q \sin \theta t$  – external dynamic load, where  $q$  – constant amplitude,  $\theta$  – the frequency of the disturbing force;  $m$  – the intensity of the distributed mass (mass per unit length) of the rod;  $y(x,t)$  – lateral movement of the point with the coordinate axis of the rod at the time (dynamic bending);  $\varphi(x,t)$  – dynamic rotation angle;  $M(x,t)$  – dynamic bending moment;  $Q(x,t)$  – dynamic shear force;  $r(x,t)$  – the intensity of the internal forces of resistance;  $f(x,t) = -m \frac{\partial^2 y}{\partial t^2}$  – the intensity of the inertial forces that arise in the course of the oscillation (the power of D'Alembert).

Assuming, according to the corrected hypothesis of Kelvin-Voigt [1, 2],

$$r(x,t) = \frac{\gamma}{\theta} EI \frac{\partial^5 y}{\partial t \partial x^4},$$

the equation of forced oscillations of the beam can be written as [1, 2]

$$EI \left( 1 + \frac{\gamma}{\theta} \frac{\partial y}{\partial t} \right) \frac{\partial^4 y}{\partial x^4} + m \frac{\partial^2 y}{\partial t^2} = q \sin \theta t, \quad (1)$$

where  $\gamma$  – inelastic resistance coefficient (dimensionless constant for a given material);  $EI$  – the bending stiffness of the beam. To this we add the equation as defined boundary conditions:

$$y(0,t) = 0; M(0,t) = 0; y(l,t) = 0; M(l,t) = 0. \quad (2)$$

In the absence of the scientific literature of the exact solution of the Eq.(1), for the study of forced oscillations of the beam based on the internal resistance to the present, as a rule, we used approximate methods. An exception may be the publication of M. Abu-Hilal [10], where the definition of the dynamic deflection of the beam is based on the method of Green's functions.

In [11] an exact solution of the Eq. (1) and fully defined dynamic parameters of the beam is found. In particular, formulas for the deflection and bending moments

$$y(x,t) = y_1(x) \sin \theta t + y_2(x) \cos \theta t \quad (3)$$

$$M(x,t) = M_1(x) (\sin \theta t + \gamma \cos \theta t) + M_2(x) (\cos \theta t - \gamma \sin \theta t), \quad (4)$$

where  $y_k(x)$ ,  $M_k(x)$  ( $k = 1, 2$ ) – the so-called constituent functions of their parameters. These functions with boundary conditions (2) at a point  $x = 0$ , defined by the formula [11]:

$$y_k(x) = (-1)^{k+1} \left( \varphi_1(0) l X_{2,k}(x) - \frac{Q_1(0) l^3}{EI} X_{4,k}(x) \right) + \varphi_2(0) l X_{2,3-k}(x) - \frac{Q_2(0) l^3}{EI} X_{4,3-k}(x) +$$

$$+ (-1)^{k+1} \frac{q l^4}{(1 + \gamma^2) EI} (X_{5,k}(x) + (-1)^k \gamma X_{5,3-k}(x));$$

$$M_k(x) = (-1)^k \left( \frac{\varphi_1(0) EI}{l} X_{4,k}^*(x) - Q_1(0) l X_{2,k}(x) \right) - \frac{\varphi_2(0) EI}{l} X_{4,3-k}^*(x) + Q_2(0) l X_{2,3-k}(x) + (-1)^k \frac{q l^2}{1 + \gamma^2} (X_{3,k}(x) + (-1)^k \gamma X_{3,3-k}(x)), \quad (6)$$

where

$$X_{n,1}(x) = \frac{1}{(n-1)!} \left( \frac{x}{l} \right)^{n-1} + \sum_{k=1}^{\infty} \frac{L^{2k} \cos k \delta}{(4k+n-1)!} \left( \frac{x}{l} \right)^{4k+n-1}, \quad (7)$$

$$X_{n,2}(x) = \sum_{k=1}^{\infty} \frac{L^{2k} \sin k \delta}{(4k+n-1)!} \left(\frac{x}{l}\right)^{4k+n-1} \quad (n = 2, 3, 4, 5), \quad (8)$$

$$X_{4,1}^*(x) = \sum_{k=0}^{\infty} \frac{L^{2k+2} \cos(k+1)\delta}{(4k+3)!} \left(\frac{x}{l}\right)^{4k+3}, \quad (9)$$

$$X_{4,2}^*(x) = \sum_{k=0}^{\infty} \frac{L^{2k+2} \sin(k+1)\delta}{(4k+3)!} \left(\frac{x}{l}\right)^{4k+3}; \quad (10)$$

$$L = \frac{\theta}{\sqrt[4]{1+\gamma^2}} l^2 \sqrt{\frac{m}{EI}}, \quad \delta = \operatorname{atan} \gamma \left(0 \leq \delta < \frac{\pi}{2}\right). \quad (11)$$

It is important to note that the parameters  $L, \delta$  and all functions (7)–(10) are dimensionless.

Constituent functions (5), (6) represented in a complex form [11]:

$$y_1(x) + iy_2(x) = (\varphi_1(0) + i\varphi_2(0))lX_2(x) - (Q_1(0) + iQ_2(0))\frac{l^3}{EI}X_4(x) + \frac{q}{(1+i\gamma)}\frac{l^4}{EI}X_5(x); \quad (12)$$

$$M_1(x) + iM_2(x) = -(\varphi_1(0) + i\varphi_2(0))\frac{EI}{l}K^2X_4(x) + (Q_1(0) + iQ_2(0))lX_2(x) - \frac{q}{(1+i\gamma)}l^2X_3(x), \quad (13)$$

where

$$X_n(x) = X_{n,1}(x) - iX_{n,2}(x) \quad (n = 2, 3, 4, 5), \\ K^2X_4(x) = X_{4,1}^*(x) - iX_{4,2}^*(x).$$

### 3.2 The dimensionless amplitude of dynamic deflections and bending moments

Unidentified initial parameters  $\varphi_1(0), \varphi_2(0), Q_1(0), Q_2(0)$  are determined from the boundary conditions (2) at the point  $x = l$ . These boundary conditions on the basis of formulas (3), (4) are equivalent to

$$y_1(l) = 0, \quad y_2(l) = 0, \quad M_1(l) = 0, \quad M_2(l) = 0,$$

and their implementation, by the formulas (12), (13), leads to a system of equations:

$$X_2(l)(\varphi_1(0) + i\varphi_2(0)) - \frac{l^2}{EI}X_4(l)(Q_1(0) + iQ_2(0)) = -\frac{ql^3}{(1+i\gamma)EI}X_5(l); \\ -\frac{EI}{l}K^2X_4(l)(\varphi_1(0) + i\varphi_2(0)) + lX_2(l)(Q_1(0) + iQ_2(0)) = \frac{ql^2}{1+i\gamma}X_3(l).$$

Hence we find the complex initial settings:

$$\varphi_1(0) + i\varphi_2(0) = \frac{ql^3}{(1+i\gamma)EI} \frac{X_3(l)X_4(l) - X_2(l)X_5(l)}{X_2^2(l) - K^2X_4^2(l)}; \quad (14)$$

$$Q_1(0) + iQ_2(0) = \frac{ql}{1+i\gamma} \frac{X_2(l)X_3(l) - K^2X_4(l)X_5(l)}{X_2^2(l) - K^2X_4^2(l)}. \quad (15)$$

To determine the required initial parameters  $\varphi_1(0), \varphi_2(0), Q_1(0), Q_2(0)$  allocate the real and imaginary terms in the right-hand sides of the formulas (14), (15). The result

$$\varphi_k(0) = \frac{ql^3}{EI(1+\gamma^2)}H_k, \quad Q_k(0) = \frac{ql}{1+\gamma^2}S_k \quad (k = 1, 2),$$

where  $H_k, S_k$  – dimensionless constants, calculated according to the formulas:

$$H_1 = \frac{a_1c_1 + a_2c_2 + \gamma(a_2c_1 - a_1c_2)}{c_1^2 + c_2^2};$$

$$H_2 = \frac{a_2c_1 - a_1c_2 - \gamma(a_1c_1 + a_2c_2)}{c_1^2 + c_2^2};$$

$$S_1 = \frac{b_1c_1 + b_2c_2 + \gamma(b_2c_1 - b_1c_2)}{c_1^2 + c_2^2};$$

$$S_2 = \frac{b_2c_1 - b_1c_2 - \gamma(b_1c_1 + b_2c_2)}{c_1^2 + c_2^2};$$

$$a_1 = X_{4,1}(l)X_{3,1}(l) - X_{4,2}(l)X_{3,2}(l) - X_{2,1}(l)X_{5,1}(l) + X_{2,2}(l)X_{5,2}(l);$$

$$a_2 = X_{2,1}(l)X_{5,2}(l) + X_{2,2}(l)X_{5,1}(l) - X_{4,1}(l)X_{3,2}(l) - X_{4,2}(l)X_{3,1}(l);$$

$$b_1 = X_{2,1}(l)X_{3,1}(l) - X_{2,2}(l)X_{3,2}(l) - X_{4,1}^*(l)X_{5,1}(l) + X_{4,2}^*(l)X_{5,2}(l);$$

$$b_2 = X_{4,1}^*(l)X_{5,2}(l) + X_{4,2}^*(l)X_{5,1}(l) - X_{2,1}(l)X_{3,2}(l) - X_{2,2}(l)X_{3,1}(l);$$

$$c_1 = X_{2,1}^2(l) - X_{2,2}^2(l) - X_{4,1}^*(l)X_{4,1}(l) + X_{4,2}^*(l)X_{4,2}(l);$$

$$c_2 = X_{4,1}^*(l)X_{4,2}(l) + X_{4,2}^*(l)X_{4,1}(l) - 2X_{2,1}(l)X_{2,2}(l).$$

Substituting the values of the initial parameters in the formula (5), (6) we have:

$$y_k(x) = \frac{ql^4}{(1+\gamma^2)EI}y_k^*(x); \quad (16)$$

$$M_k(x) = \frac{ql^2}{1+\gamma^2}M_k^*(x) \quad (k = 1, 2), \quad (17)$$

where  $y_k^*(x)$ ,  $M_k^*(x)$  – dimensionless functions, which have the form

$$y_k^*(x) = (-1)^{k+1}(H_1 X_{2,k}(x) - S_1 X_{4,k}(x) + H_2 X_{2,3-k}(x) - S_2 X_{4,3-k}(x) + (-1)^{k+1}(X_{5,k}(x) + (-1)^k \gamma X_{5,3-k}(x)),$$

$$M_k^*(x) = (-1)^k(H_1 X_{4,k}^*(x) - S_1 X_{2,k}(x) - H_2 X_{4,3-k}^*(x) + S_2 X_{2,3-k}(x) + (-1)^k(X_{3,k}(x) + (-1)^k \gamma X_{3,3-k}(x)).$$

To study the vibrations, the formula, which clearly highlighted the amplitude function, will be more convenient. Rearranging equation (3), (4) to a desired form, taking into consideration the representations (16), (17) we finally obtain:

$$y(x,t) = Am y(x,t) \sin(\theta t + \chi_y(x));$$

$$Am y(x,t) = \frac{ql^4}{(1+\gamma^2)EI} y(x);$$

$$y(x) = \sqrt{(y_1^*(x))^2 + (y_2^*(x))^2}, \chi_y(x) = \text{atan} \frac{y_2^*(x)}{y_1^*(x)};$$

$$M(x,t) = Am M(x,t) \sin(\theta t + \chi_M(x));$$

$$Am M(x,t) = \frac{ql^2}{1+\gamma^2} M(x);$$

$$M(x) = \sqrt{(M_1^*(x) - \gamma M_2^*(x))^2 + (M_2^*(x) + \gamma M_1^*(x))^2};$$

$$\chi_M(x) = \text{arctg} \frac{M_2^*(x) + \gamma M_1^*(x)}{M_1^*(x) - \gamma M_2^*(x)}.$$

As can be seen, the formula for the amplitude contains dimensionless factors  $y(x)$  and  $M(x)$ , that is independent

of the load. This factors up to size ratio  $\frac{ql^4}{(1+\gamma^2)EI}$  and

$\frac{ql^2}{1+\gamma^2}$  it defines the main forms for the dynamic deflection

and dynamic bending moment. The functions  $y(x)$  and  $M(x)$  can also be interpreted as a dimensionless amplitude.

The maximum value of the amplitude will obviously be achieved in the middle of the beam, i.e.

$$\max Am y(x,t) = \frac{ql^4}{(1+\gamma^2)EI} y\left(\frac{l}{2}\right), \tag{18}$$

$$\max Am M(x,t) = \frac{ql^2}{1+\gamma^2} M\left(\frac{l}{2}\right). \tag{19}$$

Thus, to determine the maximum amplitude, it is necessary to calculate the corresponding value of the

dimensionless amplitude in the middle of the beam. This will represent a special interest in the calculation of the resonance zones, when the frequency  $\theta$  of forced oscillations will be located in the vicinity of the frequency of free vibrations of the beam.

As is known [1], for the frequency of free oscillations of the beam, excluding the resistances, there is the formula

$$\omega_j = \frac{K_j}{l^2} \sqrt{\frac{EI}{m}} \quad (j = 1, 2, 3, \dots), \tag{20}$$

where  $K_j$  – dimensionless coefficients of free oscillations.

In the case of simply supported beam  $K_j = (j\pi)^2$ .

As the frequency of forced oscillations set the following values:

in the interval  $0 < \theta \leq \omega_1$  we believe

$$\theta = k \frac{\omega_1}{10} \quad (k = 1, 2, 3, \dots, 10);$$

in the interval  $\omega_1 < \theta \leq \omega_2$  we believe

$$\theta = \omega_1 + k \frac{\omega_2 - \omega_1}{10} \quad (k = 1, 2, \dots, 10);$$

in the interval  $\omega_2 < \theta \leq \omega_3$  we believe

$$\theta = \omega_2 + k \frac{\omega_3 - \omega_2}{10} \quad (k = 1, 2, 3, \dots, 10);$$

in the interval  $\omega_3 < \theta < \omega_4$  we believe

$$\theta = \omega_3 + k \frac{\omega_4 - \omega_3}{10} \quad (k = 1, 2, 3).$$

Substituting here instead of frequency  $\omega_1, \omega_2, \omega_3, \omega_4$  their values (16), we obtain the representation

$$\theta = \frac{K^*}{l^2} \sqrt{\frac{EI}{m}}, \tag{21}$$

where  $K^*$  – dimensionless ratio of forced oscillations for which we have

$$K^* = \begin{cases} 0,1k\pi^2, & \text{when } \theta = k \frac{\omega_1}{10}; \\ (1+0,3k)\pi^2, & \text{when } \theta = \omega_1 + k \frac{\omega_2 - \omega_1}{10}; \\ (4+0,5k)\pi^2, & \text{when } \theta = \omega_2 + k \frac{\omega_3 - \omega_2}{10}; \\ (k = 1, 2, 3, \dots, 10); \\ (9+0,7k)\pi^2, & \text{when } \theta = \omega_3 + k \frac{\omega_4 - \omega_3}{10}; \\ (k = 1, 2, 3). \end{cases}$$

When this parameter  $L$ , determined by the first of the formulas (11), taking into account (21), we obtain

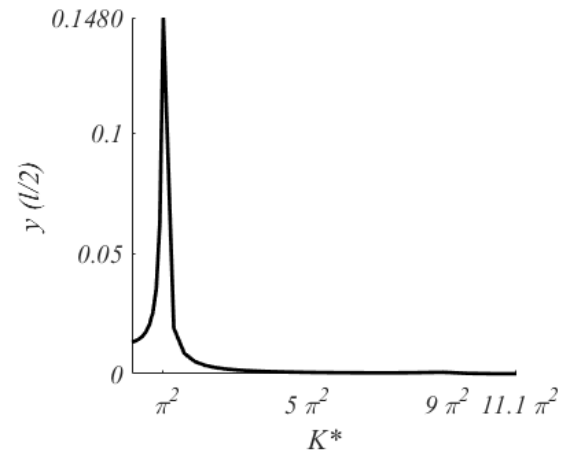
$$L = \frac{K^*}{\sqrt[4]{1 + \gamma^2}}.$$

Tab. 1 shows the results of calculations of the dimensionless deflection and bending moment amplitudes in the middle of the beam, corresponding to different values of coefficient of forced oscillation  $K^*$ . Especially note that these values are invariant relative to dimensional values  $l, EI, m, q$  and depend only on beams ends restraints. In fact, they describe the very essence of the phenomenon.

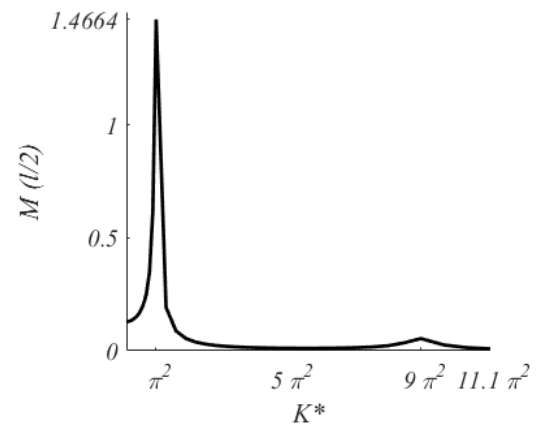
**Table 1** Maximum values of dimensionless amplitudes

| $K^*$       | For deflection | For bending moment |
|-------------|----------------|--------------------|
|             | $y(l/2)$       | $M(l/2)$           |
| $0.1\pi^2$  | 0.013204       | 0.127292           |
| $0.2\pi^2$  | 0.013614       | 0.131360           |
| $0.3\pi^2$  | 0.014358       | 0.138729           |
| $0.4\pi^2$  | 0.015546       | 0.150496           |
| $0.5\pi^2$  | 0.017393       | 0.168793           |
| $0.6\pi^2$  | 0.020338       | 0.197974           |
| $0.7\pi^2$  | 0.025397       | 0.248102           |
| $0.8\pi^2$  | 0.035476       | 0.347992           |
| $0.9\pi^2$  | 0.062745       | 0.618348           |
| $\pi^2$     | 0.148025       | 1.466401           |
| $1.3\pi^2$  | 0.018987       | 0.191677           |
| $1.6\pi^2$  | 0.008483       | 0.087690           |
| $1.9\pi^2$  | 0.005097       | 0.054210           |
| $2.2\pi^2$  | 0.003483       | 0.038293           |
| $2.5\pi^2$  | 0.002564       | 0.029264           |
| $2.8\pi^2$  | 0.001982       | 0.023597           |
| $3.1\pi^2$  | 0.001587       | 0.019803           |
| $3.4\pi^2$  | 0.001306       | 0.017153           |
| $3.7\pi^2$  | 0.001099       | 0.015250           |
| $4\pi^2$    | 0.000941       | 0.013865           |
| $4.5\pi^2$  | 0.000752       | 0.012349           |
| $5\pi^2$    | 0.000622       | 0.011532           |
| $5.5\pi^2$  | 0.000532       | 0.011248           |
| $6\pi^2$    | 0.000468       | 0.011453           |
| $6.5\pi^2$  | 0.000426       | 0.012213           |
| $7\pi^2$    | 0.000402       | 0.013748           |
| $7.5\pi^2$  | 0.000401       | 0.016610           |
| $8\pi^2$    | 0.000434       | 0.022275           |
| $8.5\pi^2$  | 0.000539       | 0.035211           |
| $9\pi^2$    | 0.000629       | 0.054233           |
| $9.7\pi^2$  | 0.000185       | 0.025754           |
| $10.4\pi^2$ | 0.000051       | 0.013671           |
| $11.1\pi^2$ | 0.000017       | 0.009066           |

Fig. 3, 4 are graphs of the maximum amplitude of the dimensionless deflection and bending moment on the coefficient of beam forced vibrations. As can be seen, the highest values of the amplitudes are achieved when the value of the coefficient of oscillations is  $\pi^2$ , which corresponds to the frequency of free oscillations  $\omega_1$ .



**Figure 3** Dependence of the dimensionless deflection amplitude of the oscillation rate



**Figure 4** Dependence of the dimensionless bending moment amplitude of the oscillation rate

Thus, according to the formulas (21), (18), (19) the calculation of the maximum amplitudes of the dynamic deflection and bending moments of the beam caused by external dynamic load  $q(x,t) = q \sin \theta t$  reduces to multiplication dimensionless value  $K^*$ ,  $y\left(\frac{l}{2}\right)$ ,  $M\left(\frac{l}{2}\right)$ , contained in Table 1, on the corresponding dimensional multipliers  $\frac{1}{l^2} \sqrt{\frac{EI}{m}}$ ,  $\frac{ql^4}{(1 + \gamma^2)EI}$ ,  $\frac{ql^2}{1 + \gamma^2}$ .

### 3.3 Example

Let us find the distribution of the amplitudes of the dynamic deflection and bending moments at the hinge beam for given values of higher frequency of forced oscillations. Inelastic material resistance factor is  $\gamma = 0.089$ . The force of inertia that occurs during the equipment operation is assumed to be  $q = 20$  kN/m, mass per unit length of the beam  $m = 2.5$  kNs<sup>2</sup>/m<sup>2</sup>, bending stiffness  $EI = 79615.11$  kNm<sup>2</sup>.

With such design data, the first four frequencies of oscillation beams excluding resistance according to (20) are equal:

$$\omega_1 = 48.9243 \text{ s}^{-1}; \omega_2 = 195.6974 \text{ s}^{-1}; \omega_3 = 440.3191 \text{ s}^{-1}; \omega_4 = 782.7895 \text{ s}^{-1}.$$

To calculate the required amplitude of dynamic deflections and bending moments corresponding to a given frequency  $\theta$ , according to the proposed method, we multiply the values that appear in the first, second and third columns of Tab. 1, respectively, on the dimensional ratios

$$\frac{1}{l^2} \sqrt{\frac{EI}{m}} = 4.9571 \text{ s}^{-1}, \quad \frac{ql^4}{(1+\gamma^2)EI} = 0.3230 \text{ m},$$

$$\frac{ql^2}{1+\gamma^2} = 714.3417 \text{ kNm}.$$

Results are shown in Tabs. 2 and 3. For comparison, the results of calculations for a given beam finite element method are obtained using ANSYS [12] software package.

**Table 2** Maximum values of dimensioned amplitudes

| Frequencies<br>$\theta, \text{ s}^{-1}$ | For deflection, m   |          |
|---|---------------------|----------|
|   | The author's method | ANSYS    |
| 4.8924                                  | 0.004265            | 0.004312 |
| 9.7849                                  | 0.004398            | 0.004402 |
| 14.6773                                 | 0.004638            | 0.004683 |
| 19.5697                                 | 0.005022            | 0.005074 |
| 24.4622                                 | 0.005618            | 0.005691 |
| 29.3546                                 | 0.006569            | 0.006611 |
| 34.2470                                 | 0.008204            | 0.008316 |
| 39.1395                                 | 0.011459            | 0.011700 |
| 44.0319                                 | 0.020267            | 0.020211 |
| $\omega_1 = 48.9243$                    | 0.047813            | 0.041122 |
| 63.6016                                 | 0.006133            | 0.005918 |
| 78.2789                                 | 0.002740            | 0.002212 |
| 92.9562                                 | 0.001646            | 0.001533 |
| 107.6336                                | 0.001125            | 0.001094 |
| 122.3109                                | 0.000828            | 0.000800 |
| 136.9882                                | 0.000640            | 0.000637 |
| 151.6655                                | 0.000513            | 0.000510 |
| 166.3428                                | 0.000422            | 0.000412 |
| 181.0201                                | 0.000355            | 0.000349 |
| $\omega_2 = 195.6974$                   | 0.000304            | 0.000293 |
| 220.1595                                | 0.000243            | 0.000213 |
| 244.6217                                | 0.000201            | 0.000180 |
| 269.0839                                | 0.000172            | 0.000143 |
| 293.5461                                | 0.000151            | 0.000139 |
| 318.0082                                | 0.000137            | 0.000111 |
| 342.4704                                | 0.000130            | 0.000100 |
| 366.9326                                | 0.000129            | 0.000099 |
| 391.3947                                | 0.000140            | 0.000100 |
| 415.8569                                | 0.000174            | 0.000114 |
| $\omega_3 = 440.3191$                   | 0.000203            | 0.000112 |
| 474.5661                                | 0.000060            | 0.000058 |
| 508.8132                                | 0.000017            | 0.000013 |
| 543.0602                                | 0.000005            | 0.000005 |

**Table 3** Maximum values of dimensioned bending moments amplitudes

| Frequencies<br>$\theta, \text{ s}^{-1}$ | For bending moment, kNm |           |
|---|-------------------------|-----------|
|   | The author's method     | ANSYS     |
| 4.8924                                  | 90.9303                 | 90.9393   |
| 9.7849                                  | 93.8362                 | 93.8732   |
| 14.6773                                 | 99.0996                 | 99.1469   |
| 19.5697                                 | 107.5053                | 107.9006  |
| 24.4622                                 | 120.5757                | 120.9934  |
| 29.3546                                 | 141.4211                | 142.1433  |
| 34.2470                                 | 177.2297                | 180.0014  |
| 39.1395                                 | 248.5850                | 256.4465  |
| 44.0319                                 | 441.7116                | 480.9984  |
| $\omega_1 = 48.9243$                    | 1047.5114               | 1210.1123 |
| 63.6016                                 | 136.9229                | 134.4555  |
| 78.2789                                 | 62.6410                 | 60.9702   |
| 92.9562                                 | 38.7246                 | 38.3356   |
| 107.6336                                | 27.3544                 | 26.7447   |
| 122.3109                                | 20.9048                 | 20.8733   |
| 136.9882                                | 16.8566                 | 16.8341   |
| 151.6655                                | 14.1464                 | 14.0050   |
| 166.3428                                | 12.2530                 | 12.0405   |
| 181.0201                                | 10.8938                 | 10.9666   |
| $\omega_2 = 195.6974$                   | 9.9044                  | 9.8888    |
| 220.1595                                | 8.8213                  | 9.0403    |
| 244.6217                                | 8.2375                  | 8.7641    |
| 269.0839                                | 8.0346                  | 8.7445    |
| 293.5461                                | 8.1815                  | 11.2345   |
| 318.0082                                | 8.7243                  | 13.1477   |
| 342.4704                                | 9.8206                  | 19.2345   |
| 366.9326                                | 11.8649                 | 28.4332   |
| 391.3947                                | 15.9119                 | 45.3221   |
| 415.8569                                | 25.1526                 | 91.3576   |
| $\omega_3 = 440.3191$                   | 38.7406                 | 111.3787  |
| 474.5661                                | 18.3970                 | 76.3744   |
| 508.8132                                | 9.7656                  | 41.3222   |
| 543.0602                                | 6.4760                  | 26.8564   |

#### 4 CONCLUSIONS

The maximum values of the amplitudes of the dimensionless dynamic deflections and bending moments are calculated. Analysis of the results shows that the sharp increase in the amplitude values of deflections and dynamic bending moments occur in the resonance regions corresponding to the first natural frequency of oscillation. At higher frequencies this effect is practically absent. At frequencies of the external load, close to the third natural frequency, there is quite a significant difference in the results obtained by the author's and the finite element method. Values obtained by the author, should be regarded as accurate, since they are obtained by the exact solution of the original differential equation by the method of direct integration.

#### 5 REFERENCES

- [1] Babakov I. M. Teoriya kolebanii / I. M. Babakov. – M.: Nauka, 1976. – 592 p.
- [2] Vasilenko M. V. Teoriya kolyvan' i stiykosti ruhu / M. V. Vasilenko, O. M. Alekseychuk. – K.: Vyscha shkola, 2004. – 525 p.
- [3] Krylov A. N. Sobranie trudov. Tom 10: Vibratsii sudov. / A.N. Krylov. – M.: Izd-vo AN USSR, 1948. – 402 p.

- [4] Timoshenko S. P. *Kolebaniya v inzhenernom dele*. M.: Nauka, 1967. – 444 p.
- [5] Panovko Y. G. *Vnutrennee trenie pri kolebaniyakh uprugikh sistem.* / Y.G. Panovko. – M.: Fizmatgiz, 1960. – 193 p.
- [6] Sorokin E. S. *K teorii vnutrennego treniya pri kolebaniyakh uprugikh sistem.* – M.: Gosstroyizdat, 1960. – 131 p.
- [7] Olshanskiy V. P. Nestacionarnie kolebaniya mekhanicheskoy sistemi lineyno-peremennoy massy s kombinirovannim treniem / V.P. Olshanskiy, S.V. Olshanskiy // *Visnyk Natsionalnogo tehnicnogo universitetu "Kharkivskiy politehnicniy institute"*. – 2013. – №4 (1041). – P. 324–333.
- [8] Olshanskiy V. P. Nestacionarnie kolebaniya oscilatora peremennoy massi s uchetom viazkogo treniya / V. P. Olshanskiy, S. V. Olshanskiy // *Vibratsii v tekhnitsi ta tekhnologiyakh.* – 2014. – Vyp. № 3 (75). – P. 18–27.
- [9] Berendeev N. N. *Issledovaniye vliyaniya vnutrennego treniya i sposoba vzbuzhdeniya na vynuzhdennie kolebaniya sistemy* / Elektronnoe uchebno-metodicheskoye posobie. – Nizhniy Novgorod: Nizhegorodskiy universitet, 2012. – 50 p.
- [10] Abu-Hilal M. Forced vibration of Euler–Bernoulli beams by means of dynamic Green functions. // *Journal of Sound and Vibration.* 267, (2003), pp. 191–207.
- [11] Krutiy Yu. S. Forced harmonic oscillations of the Euler–Bernoulli beam with resistance forces: Odeskyi Politehnicnyi Universytet. Pratsi. Machine building, process metallurgy, materials science / Yu. S. Krutiy // *Pratsi OPU.* 3(47)(2015), pp. 9–16.
- [12] Daschenko A. F. ANSYS v zadachakh inzhenernoi mekhaniki / A. F. Daschenko, D. V. Lazareva, N. G. Surianinov / *Izd. 2-e, pererab. i dop. Pod red. N. G. Surianinova.* – Odesa. – Palmira, 2011. – 505 p.

**Authors' contacts:****Yurii Krutii,**

Odesa State Academy of Civil Engineering and Architecture  
Didrikhsona str. 4, 65029 Odesa, Ukraine

**Nikolay Suryaninov**

Odesa State Academy of Civil Engineering and Architecture  
Didrikhsona str. 4, 65029 Odesa, Ukraine  
shylyiaiev@gmail.com