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Connection of Solutions of Abstract Paired Equations in Rings with Factorization Pairs

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Abstract. The connection between solutions of the abstract paired equations with respect to an unknown $x \in (R_1 \cap R_2)$:

$$\begin{cases} (a_1 x)^- &= c^-, \\ (a_2 x)^+ &= b^+, \end{cases}$$

is considered. It is assumed that the coefficients $a_j \in R_j$, $j = 1, 2$, where R_1, R_2 are the associative rings with factorization pairs (R_j^+, R_j^-) and unity $e \in R_1 \cap R_2$.

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A connection between solutions of abstract paired equations with an arbitrary right-hand side and the same equations with the multiplicative unity of a ring in right-hand side is analyzed. The cases when the coefficients belong to the same or different rings with factorization pairs are considered. The possibility of application to paired integral and paired matrix equations is pointed out.

The appearance of the notion of rings with factorization pairs [1–6] and the paired equations in the forms

$$\begin{cases} (a_1 x)^- &= c^- \\ (a_2 x)^+ &= b^+, \end{cases} \tag{1}$$

is connected with the penetration of Banach algebra ideas into the theory of convolution type integral equations, which was initiated by M.G. Krein [7]. The equations of type (1) are found in investigations of integral equations with kernels depending on the difference of the arguments, including the paired equations of the convolution type:

$$\begin{cases} \varphi(t) - \int\limits_{-\infty}^{\infty} k_1(t-s)\varphi(s)\,ds &= f(t), \qquad -\infty < t < 0, \\ \varphi(t) - \int\limits_{-\infty}^{\infty} k_2(t-s)\varphi(s)\,ds &= f(t), \qquad 0 < t < \infty; \end{cases} \tag{2}$$

where $k_j(t)\exp\{c_j t\} \in L_1(-\infty, \infty)$, $c_j \in \mathbb{R}$, $j = 1, 2$, and also in special applied tasks [1, 3, 7 – 11].

The equations (1) are subtypes of the general form of paired equations:

$$\begin{cases} [a_{11}\, x\, a_{12}]^- & = & c^- \\ [a_{21}\, x\, a_{22}]^+ & = & b^+, \end{cases} \tag{3}$$

with respect to an unknown element x [2, 3, 11]. The abstract paired equations (1), (3) with factorizable coefficients in rings with factorization pairs were, in particular, studied in [3, 11]. The results concerning the solvability of these equations through the factorization of elements which may be built with the help of coefficients were presented during the international conferences [5, 12] and others.

This article is devoted to the investigation of connection between solutions of the abstract paired equations (1) with respect to the unknown element x.

1. Notations, definitions, and general provisions

1.1. Following [3, 4, 6, 13], by R we shall denote any, in general, non-commutative, and, probably, non-associative ring with unity e. Let p^+, p^- be commutative projectors, i. e. additive and idempotent mappings from R into R. Let us assume: $p^0 := p^+p^- (= p^-p^+)$; $p_\mp := p^\mp - p^0$. For any subset $B \subseteq R$ we shall denote: $B^{\mp,0} := p^{\mp,0}B$; $B_\mp := p_\mp B$; $B^* := B^+ + B^-$; $B_* := B_+ + B_-$. For any $x \in R$ we note $x^{\mp,0} := p^{\mp,0}x$; $x_\mp := p_\mp x$. The inverse in R of an element $x \in R$ invertible in R will be denoted by the symbol x', if necessary, additionally supplied. For any subsets $A, B \subseteq R$ we shall define the set $\mathrm{inv}(A, B) := \{x \in A, x'$ exists and belongs to $B\}$. Let us denote $\mathrm{inv}(A, A) := \mathrm{inv}A$. The element u^+, [the element v^0, the element w^-] will be called *correct* [6], if $u^+ \in \mathrm{inv}R^+$, [$v^0 \in \mathrm{inv}R^0$, $w^- \in \mathrm{inv}R^-$].

1.2. Supplementing [2,3,13], where, in particular, the concept of factorization of structures [6] is developed, and [8], we shall introduce the following definitions.

Definition 1. *A pair of subrings (R^+, R^-) $[\equiv (R^-, R^+)]$ in ring R with unity e will be called the left factorization pair (LFP) of the ring R, if there are commutative projectors p^+, p^- generated which act on $R^\mp := p^\mp R$ and satisfy the following axioms:*

$$e \in R^0; \tag{4}$$

$$p^0 \textit{ is a ring homomorphism from } R^+ \textit{ and } R^- \textit{ into } R^0; \tag{5}$$

$$R^+R^- \subseteq R^*. \tag{6}$$

The right factorization pair (RFP) [2] is defined similarly. It should be noted that factorizations of structures in R [6] correspond here to LFP of R. If R is commutative and whenever the pair (R^+, R^-) is simultaneously LFP and RFP of R, this pair will be called a factorization pair (FP) of the ring R.

Definition 2. *Any ring R with unity e, considered together with its fixed FP (R^+, R^-) $[\equiv (R^-, R^+)]$ will be called a ring with factorization pair.*

Nontrivial examples of rings with FP can be constructed by starting, for example, from rings of matrices, rings of absolute integrable functions, their appropriate transformations, and others [3, 4, 6, 7, 10, 13].

1.3. We shall say ([3], compare with [6]) that an element $a \in R$ allows in R a left factorization, l.f., (right factorization, r.f.) by a pair (R^+, R^-) if there are elements $r^+ \in R^+$, $s^0 \in R^0$, $t^- \in R^-$ such that

$$a = r^+ s^0 t^-, \qquad (a = t^- s^0 r^+).$$

The multipliers $r^+ \in R^+$, $s^0 \in R^0$, $t^- \in R^-$ are referred to as plus-, diagonal- and minus-factors, respectively. An l.f. (r.f.) is referred to as correct left factorization, c.l.f., (correct right factorization, c.r.f.) if $r^+ \in R^+$, $s^0 \in R^0$, $t^- \in R^-$ are correct elements; – as normalized left factorization, n.l.f., (normalized right factorization, n.r.f.) if $t^0 = r^0 = e$, and as normalized correct left factorization, n.c.l.f., (normalized correct right factorization, n.c.r.f.) if it is both c.l.f. (c.r.f.) and $t^0 = r^0 = e$.

1.4. When the problem of solvability of the abstract paired equations (1), (3) is posed in a ring R with the factorization pair (R^+, R^-), an element $x \in R^*$ will be considered as unknown, and the rest of the elements will be assumed to be given. We also assume that $c^- \in R^-$, $b^+ \in R^+$ and the coefficients a_1, a_2, a_{11},a_{12},a_{21},a_{22} are invertible in R. It is assumed that a solution to the paired equation (1) is an element $x \in R$ such that the corresponding right-hand and left-hand sides of equations (1) coincide after x is substituted into them.

The problem of solvability of paired equation (1) in $R_{1\cap 2} := R_1 \cap R_2$ in the case where the coefficients $a_i \in R_i$; $i = 1, 2$, R_1 and R_2 are the rings with factorization pairs, with unity $e \in R_{1\cap 2}$ and common multiplication [3], may be posed as well. In this case, we define $R_{1\cup 2} := (R_1 + R_2)$.

Below, we shall assume that the following conditions hold:

$$R_1^- \subseteq R_2^-, \qquad R_1^+ \supseteq R_2^+. \tag{7}$$

2. Main result

2.1. Solution $x_e \in R^*_{1\cap 2}$ of abstract paired equation (1) with coefficients a_i; $i = 1, 2$ invertible in their rings R_i for the right-hand side $c^- = b^+ = e$ plays a special role in the theory of solvability of these equations. Under some conditions, the solution $x \in R^*_{1\cap 2}$ of (1) with arbitrary right-hand part $c^- \in R_1^-$, $b^+ \in R_2^+$ can be expressed through it.

Theorem 1. (*Connection of solutions*). *Let R_1, R_2 be associative and, in general, non-commutative rings with common multiplication, common unity $e \in R_{1\cap 2}$, and FP (R_j^+, R_j^-) generated by commutating projectors p^+, p^-: $R_{1\cup 2} \to R_{1\cup 2}$, so that conditions* (7) *hold true, and the coefficients of equations* (1) *$a_j \in invR_j^*$; $j = 1, 2$.*

Let the abstract paired equation:

$$\begin{cases} (a_1 x_e)^- &= e, \\ (a_2 x_e)^+ &= e, \end{cases} \tag{8}$$

have a solution $x_e \in R^*_{1\cap 2}$ *in* $R_{1\cap 2}$ *which has the inverse*

$$[x_e]'_{R_1^*} := [x_e]'_{R_1} \in R_1^*,$$

$$[x_e]'_{R_2^*} := [x_e]'_{R_2} \in R_2^*.$$

Then for any right-hand part (c^-, b^+), $c^- \in R_1^-$, $b^+ \in R_2^+$, *by the compatibility condition*

$$[(a_1 x_e)'_{R_1^*} c^-]^0 = [(a_2 x_e)'_{R_2^*} b^+]^0, \tag{9}$$

the solution $x \in R^*_{1\cap 2}$ *of (1) in* $R_{1\cap 2}$ *can be represented as*

$$x = x_e\{[(a_2 x_e)'_{R_2^*} b^+]^+ + [(a_1 x_e)'_{R_1^*} c^-]_-\}. \tag{10}$$

[Here, $(a_j x_e)'_{R_j^*}$ *are the inverses of* $a_j x_e$ *in* R_j*, which exist and belong to* R_j^**;* $j = 1, 2$ *under the assumptions of Theorem 1.]*

Proof. Under the assumptions of Theorem 1 and for any $c^- \in R_1^-$, $b^+ \in R_2^+$, the right-hand side of formula (10) is meaningful and is an element $x \in R^*_{1\cap 2}$. Substituting this element x into the left-hand side of the paired equation (1) and transforming it with the help of ring operations acting in R_j; $j = 1, 2$, and projectors p^+, p^-, one can be convinced that it satisfies the equation (1) if the compatibility condition (9) holds. Indeed,

$$\begin{aligned} (a_1 x)^- &= (a_1 x_e\{[(a_2 x_e)'_{R_2^*} b^+]^+ + [(a_1 x_e)'_{R_1^*} c^-]_-\})^- \\ &= \{(a_1 x_e)^+[(a_2 x_e)'_{R_2} b^+]^+\}^0 + (a_1 x_e\{(a_1 x_e)'_{R_1} c^- - [(a_1 x_e)'_{R_1} c^-]^+\})^- \\ &= (a_1 x_e)^0[(a_2 x_e)'_{R_2} b^+]^0 + [a_1 x_e (a_1 x_e)'_{R_1} c^-]^- - \{(a_1 x_e)^+[(a_1 x_e)'_{R_1} c^-]^+\}^0 \\ &= [(a_2 x_e)'_{R_2} b^+]^0 + c^- - [(a_1 x_e)'_{R_1} c^-]^0 \\ &= c^-. \end{aligned}$$

Similarly, it can be proved that

$$(a_2 x)^+ = (a_2 x_e\{[(a_2 x_e)'_{R_2^*} b^+]^+ + [(a_1 x_e)'_{R_1^*} c^-]_-\})^+ = b^+.$$

Theorem is proved. □

2.2. If $R_1 = R_2 = R$, then from Theorem 1 it follows

Theorem 2. *Let* R *be an associative and, in general, non-commutative ring with unity* e *and FP* (R^+, R^-) *generated by commutating projectors* $p^+, p^- : R \to R$*. Let* $a_j \in invR^*$ *be the coefficients and there exists a solution* $x_e \in invR^*$ *of the paired equation (8) invertible in* R*. Then for any right-hand part satisfying the compatibility condition*

$$[(a_1 x_e)'_{R^*} c^-]^0 = [(a_2 x_e)'_{R^*} b^+]^0,$$

one of the solutions $x \in R^*$ of (1) has the representation

$$x = x_e\{[(a_2 x_e)'_{R^*} b^+]^+ + [(a_1 x_e)'_{R^*} c^-]_-\}. \tag{11}$$

In this formula, $(a_j x_e)'_{R^*}$, $j = 1, 2$, denotes the corresponding inverse element in R.

Let us point out that the solutions x_e, having the required inverses, exist, in particular, under corresponding correct factorizations by the factorization pairs of the ring of the elements, which defined by the coefficients and also in some more general situations.

2.3. For example, if $R = R_1 = \widetilde{L}$, $R_2 = \widetilde{L}_{<c>}$; $c \geqslant 0$ [2, 3, 10], then theorems 1, 2 can be applied to paired integral equations of the convolution type (2), in particular, with kernel functions $k_{1,2}(t) \in L_1(-\infty, \infty)$ or $k_1(t), k_2(t)\exp(ct) \in L_1(-\infty, \infty)$, $c \geqslant 0$ [8–10].

When $R = R_{n\times n}$; $n \geqslant 2$ [6, 14], from theorem 2 it follows the formula of connection of solutions of the corresponding paired matrix equations with unknown matrix $X \in R_{n\times n}$ and projectors $p^+, (p^-) : R_{n\times n} \to R_{n\times n}$, which map each matrix from $R_{n\times n}$ onto the corresponding lower (upper) triangular:

$$\begin{cases} [A_1 X]^- &= C^- \\ [A_2 X]^+ &= B^+; \end{cases} \tag{12}$$

where $A_1, A_2 \in R_{n\times n}$, $C^- \in R^-_{n\times n}$, $b^+ \in R^+_{n\times n}$ are given matrices. Thus, C^- and B^+ are right- and left-triangular matrices from $R_{n\times n}$ respectively.

2.4. Note that for the abstract paired equation (1) in rings with factorization pairs the connection between the solutions corresponding to $c^- := a_{1-}$ and $b^+ := a_{2+}$ is established as well.

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