Journal of the Egyptian Mathematical Society (2013) 21, 206–212

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

The unbounded solution of a periodic mixed Sturm–Liouville problem in an infinite strip for the Laplacian

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Received 13 June 2012; revised 12 November 2012; accepted 21 April 2013
Available online 13 June 2013

KEYWORDS
Mixed Sturm–Liouville problem;
Fourier transform;
Integral equations

Abstract The unbounded solution, at the points where the boundary conditions change, for a mixed Sturm–Liouville problem of the Dirichlet–Neumann type can be obtained using the method of the integral equation formulation. Since this formulation is usually reduced to an infinite algebraic system in which the unknowns are the Fourier coefficients of the unknown unbounded entity, a study of ℓ_p-solutions imposes itself concerning the influence of the truncation on such systems. This study is achieved and the well-known theorem on the ℓ_2-solutions of the infinite algebraic systems is generalized.

MATHEMATICS SUBJECT CLASSIFICATION:
54xx
42A38
34B24

1. Introduction

The formulation of mixed Sturm–Liouville problems [1–4] into a Cauchy type integral equation can be successfully applied to just obtain the solutions corresponding to which the necessarily continuous quantities, like temperature or displacement, etc., are Hölder-continuous at the points where the boundary conditions change. On the other hand, at such points, quantities like heat flow and normal contact stress, for example, can physically as well as mathematically [5] be unbounded. To obtain the solutions which designate these interesting singulari-

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Peer review under responsibility of Egyptian Mathematical Society.

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http://dx.doi.org/10.1016/j.joems.2013.04.006

ties, the method used in the above mentioned works is inconvenient for two reasons: The Hölder-continuity imposed on the restriction of the solution on the boundary, to permit there a Fourier representation for other quantities, may throw out solutions that involve singularities. Further, the unknown function of the integral equations used therein represents a continuous quantity, the sole function which may be obtained in a closed form. The other quantities in the problem can be obtained as restrictions of a series and the singularities, if any, are practically lost.

In this work we show that the physically important unbounded solutions of Sturm–Lioville problems can also be obtained by means of the integral equation formulation. This is achieved by carrying out a modification so that the problem is converted to a Hilbert-type integral equation in which the unknown function represents quantities that may be unbounded. In this way the form of the singularity can be definitely obtained. The typical example considered here is the periodic Sturm–Lioville problem in a rectangular region subjected at one edge to a Dirichlet–Neumann condition.

In Section 2 we outline the procedures of the formulation. The complex Fourier transform

$$F_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx, \quad f(x) = \sum_{n=-\infty}^{\infty} F_n\, e^{inx}$$

is used to reduce the problem to a discrete problem and in turn to a Hilbert-equation which is finally reduced to an infinite system of algebraic equations. The solution(s) of the later system may be found in $\ell_p (p > 4)$.

In Section 3 we prove that to any eigenvalue of the problem there corresponds a unique solution of the algebraic system. To this end we studied the ℓ_p-solutions of the infinite algebraic systems.

Numerical verifications are given in the fourth section, while the fifth and last section is devoted to summarize the results of this paper.

2. A typical example and its solution

The problem we shall use here to present the procedures is to search for the even 2π periodic solutions of the problem

$$\Delta u + \gamma^2 u = 0, \quad \text{in } \Omega \subset R^2 \tag{2.1}$$

subjected to the mixed boundary conditions:

$$u = 0, \quad \text{on } \Gamma_+ \cup \Gamma \tag{2.2}$$

$$\frac{\partial u}{\partial y} = 0, \quad \text{on } \Gamma_- \tag{2.3}$$

where

$$\Omega = (-\pi, \pi) \times (0, 1), \quad \Gamma_+ = \{(x, 0); |x| < c\},$$
$$\Gamma_- = \{(x, 0); (-\pi, \pi) - \Gamma_-\}, \quad \text{and } \Gamma = \{(x, 1), |x| \leqslant \pi\}.$$

The mixed boundary condition imposed in the lower side $y = 0$ according to Eqs. (2.2) and (2.3) can be replaced by the two uniform and compatible ones:

$$u = \varphi_+(x;\gamma) = \begin{cases} 0, & \text{on } \Gamma_+ \\ \text{undetermined}, & \text{on } \Gamma_- \end{cases} \tag{2.4}$$

and

$$\frac{\partial u}{\partial y} = \varphi_-(x;\gamma) = \begin{cases} \text{undetermined}, & \text{on } \Gamma_+ \\ 0, & \text{on } \Gamma_- \end{cases} \tag{2.5}$$

The restriction

$$\frac{\partial u(x, 0; \gamma)}{\partial y} = \varphi_-(x;\gamma) \text{ on } \Gamma_- \cup \Gamma_+ \tag{2.6}$$

for all possible values of γ belonging to the class of the integrable functions. On Applying the finite Fourier transform with respect to x to Eq. (2.1), the solution of the resulting equation is

$$U_n(y) = A_n e^{k_n y} + B_n e^{-k_n y},$$

where

$$k_n = \begin{cases} i\sqrt{\gamma^2 - n^2}, & |n| < \gamma, \\ \sqrt{n^2 - \gamma^2}, & |n| > \gamma. \end{cases} \tag{2.7}$$

Substituting this solution in the Fourier transform of the boundary conditions (2.2) and (2.3) and then eliminating the constants A_n and B_n, we arrive at the following discrete problem [6] (see also the Refs. [1–4])

$$R_n(\gamma)\Phi_{n-}(\gamma) = -\Phi_{n+}(\gamma),$$

where

$$R_n(\gamma) = \frac{1}{k_n}\tanh k_n, \quad \text{and } \Phi_{n\pm} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_\pm(x) e^{-inx} dx. \tag{2.8}$$

The solutions of problem (2.1) along with condition (2.5) are then

$$\begin{aligned} u(x, y; \gamma) = &-\frac{\sin(\gamma(1-y))}{\gamma\cos\gamma}\Phi_{0-}(\gamma) \\ &- 2\sum_{n<|\gamma|}\frac{\sin\left(\sqrt{\gamma^2-n^2}(1-y)\right)\cos(nx)}{\sqrt{\gamma^2-n^2}\cos\sqrt{\gamma^2-n^2}}\Phi_{n-}(\gamma) \\ &- 2\sum_{n>|\gamma|}\frac{\sinh\left(\sqrt{n^2-\gamma^2}(1-y)\right)\cos(nx)}{\sqrt{n^2-\gamma^2}\cosh\sqrt{n^2-\gamma^2}}\Phi_{n-}(\gamma), \end{aligned} \tag{2.9}$$

where γ stands for any possible value at which system (2.8) has a solution $\Phi_{n\pm}(\gamma); n \in \mathbb{N}^+$. The discrete problem (2.8) can be rewritten in the more convenient form:

$$\text{sgn}\left(n + \frac{1}{2}\right)\Phi_{n-}(\gamma) + \Gamma_n(\gamma)\Phi_{n-}(\gamma) = -n\Phi_{n+}(\gamma), \quad n \in Z \tag{2.10}$$

where

$$\Gamma_n(\gamma) = nR_n(\gamma) - \text{sgn}\left(n + \frac{1}{2}\right) = O\left(\frac{1}{n^2}\right). \tag{2.11}$$

Strictly speaking, problem (2.10) relates between the derivatives of both functions $\varphi_\pm(x;\gamma)$. The n^{th} Fourier component of a derivative is equal to *in* times the same component of its antiderivative. Thus the coefficients $\Phi_{0\pm}(\gamma)$ are lost in problem (2.10), and it does not provide with the connection between them and the other components of the unknown functions $\varphi_\pm(x;\gamma)$. It is easily seen that the equation in problem (2.10) corresponding to $n = 0$ is trivial. Applying the inverse Fourier transform

$$\varphi_{\pm}(x;\gamma) = \sum_{n=-\infty}^{n=\infty} \Phi_{n\pm}(\gamma)e^{inx}$$

to the discrete problem (2.10), using the relation [6]

$$\sum_{n=-\infty}^{n=\infty} \operatorname{sgn}\left(n+\frac{1}{2}\right)\Phi_{n-}e^{inx} = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\varphi_-(t;\gamma)e^{it}}{e^{it}-e^{ix}}dt$$

and taking into account that $\varphi_+(x;\gamma) = 0$ on Γ_+, the discrete problem (2.10) is reduced to

$$\frac{1}{\pi}\int_{-c}^{c} \frac{\varphi_-(t;\gamma)e^{it}}{e^{it}-e^{ix}}dt + \sum_{n=-\infty}^{n=\infty} \Gamma_n\Phi_{n-}(\gamma)e^{inx} = 0, \ x \in \Gamma_+ \qquad (2.12)$$

but since

$$\frac{e^{it}}{e^{it}-e^{ix}} = \frac{1}{2}\left(1 - i\cot\left(\frac{t-x}{2}\right)\right), \qquad (2.13)$$

we get

$$\Phi_{0-}(\gamma) - \frac{i}{2\pi}\int_{-c}^{c} \cot\left(\frac{t-x}{2}\right)\varphi_-(t;\gamma)dt$$
$$= \Phi_{0-}(\gamma) - 2i\sum_{n=1}^{\infty}\Gamma_n\Phi_{n-}(\gamma)\sin nx.$$

Again, the explicit zero-order coefficient $\Phi_{0-}(\gamma)$ is deleted, and we finally have

$$\frac{1}{2\pi}\int_{-c}^{c} \cot\left(\frac{t-x}{2}\right)\varphi_-(t;\gamma)\,dt$$
$$= 2\sum_{n=1}^{\infty}\Gamma_n(\gamma)\Phi_{n-}(\gamma)\sin(nx); \quad x \in (-c, c). \qquad (2.14)$$

The Hilbert-type equation (2.14) can be inverted into the class of integrable functions [7], with the result

$$\varphi_-(x;\gamma) = \frac{1}{X(x)}\left(a_0\cos\frac{x}{2} - 2\sum_{n=1}^{\infty}\Gamma_n(\gamma)\Phi_{n-}(\gamma)V_n(x)\right), \qquad (2.15)$$

where the constant a_0 reflects the fact that this equation is sufficient to determine $\varphi_-(x;\gamma)$ only to within an additive constant,

$X(x) = \sqrt{2(\cos x - \cos c)}$, and

$$V_n(x) = \frac{1}{2\pi}\int_{-c}^{c}\frac{X(t)\sin nt}{\sin\left(\frac{t-x}{2}\right)}dt. \qquad (2.16)$$

The explicit expression of the integrals $V_n(x)$ are

$$V_n(x) = \sum_{m=0}^{n}\mu_{n-m}(\cos c)\cos\left(m+\frac{1}{2}\right)x, \ \mu_0(\cos c) = 1,$$
$$\mu_1(\cos c) = -\cos c \qquad (2.17)$$

and $\mu_k(\cos c) = \dfrac{P_{k-2}(\cos c) - P_k(\cos c)}{2k-1}, \ k \geqslant 2,$

where $P_n(\cos c)$ are the Legendre polynomials

$$P_n(\cos c) = \frac{1}{\pi}\int_{-c}^{c}\frac{\cos\left(n+\frac{1}{2}\right)x}{X(x)}dx. \qquad (2.18)$$

The application of the finite Fourier transform to Eq. (2.15) leads to the algebraic system

$$\Phi_{\ell-}(\gamma) = a_0N_{0\ell} - 2\sum_{n=1}^{\infty}\Gamma_n(\gamma)N_{n\ell}\Phi_{n-}(\gamma), \ \ell \in \mathbb{N}^+, \qquad (2.19)$$

where

$$N_{n\ell} = \frac{1}{2\pi}\int_{-c}^{c}\frac{V_n(x)\cos(\ell x)}{X(x)}dx$$
$$= \frac{1}{4}\sum_{m=0}^{n}\mu_{n-m}(\cos c)(P_{m-\ell}(\cos c) + P_{m+\ell}(\cos c)), \quad \ell \in \mathbb{N}^+, \qquad (2.20)$$

and

$$N_{0\ell} = \frac{1}{2\pi}\int_{-c}^{c}\frac{\cos(\ell x)\cos(x/2)}{X(x)}dx$$
$$= \frac{1}{4}(P_\ell(\cos c) + P_{\ell-1}(\cos c)) \quad \ell \in \mathbb{N}^+. \qquad (2.21)$$

These relations hold for $c \in (0, \pi)$. In contrast to what we have emphasized above the first equation of system (2.19) defines $\Phi_{0-}(\gamma)$ in terms of the other components to within an additive constant and this does not represent a contradiction if this system is reducible. Indeed, from definition (2.20)

$$N_{n0} = 0, n \in \mathbb{N}, \qquad (2.22)$$

and hence

$$a_0 = 2\Phi_0. \qquad (2.23)$$

Thus system (2.19) is reduced to

$$\Phi_{\ell-}(\gamma) = 2\Phi_0N_{0\ell} - 2\sum_{n=1}^{\infty}\Gamma_n(\gamma)N_{n\ell}\Phi_{n-}(\gamma), \quad \ell \in \mathbb{N}. \qquad (2.24)$$

Again, system (2.24) lacks information about the component $\Phi_{0-}(\gamma)$. Indeed the designation of the values of γ at which its truncation at any order N may have solutions, i.e. the zeroes of the corresponding determinant, implies the consideration of an additional component $\Phi_{(N+1)-}(\gamma)$ since this determinant is of order $N+1$. A quantity which was preliminary excluded cannot be restored merely through the frame of the method or its self-consistency. In order to repair this erratic attitude we return to the original problem from which it is clear that the condition $u(0,0,\gamma) = 0$ is inevitable as long as $c > 0$. Expressed in terms of the solution (2.9) this yields

$$R_0(\gamma)\Phi_{0-}(\gamma) + 2\sum_{n=1}^{\infty}R_n(\gamma)\Phi_{n-}(\gamma) = 0 \qquad (2.24')$$

which is an alternative equation to that lost on multiplying the first equation of discrete problem (2.8) by zero. By the truncation of the infinite algebraic system at order N we mean that obtained respectively from (2.24′) together with the first N equations of (2.24). As in Refs. [3,4], it is a simple matter to verify that if γ is a zero of the determinant of the algebraic system truncated at order N, then its corresponding solution together with $\Phi_{n-} = 0, n > N+1$, defines through (2.15) an exact solution of Eq. (2.12) truncated such that $|n| < N+1$,

Table 1 The first eigenvalues obtained at different orders of truncation.

i / N	1	2	3	4	5
10	1.79428	2.93754	3.99696	4.73586	5.36158
20	1.79422	2.93900	3.99837	4.73543	5.35902
30	1.79422	2.93856	3.99805	4.73540	5.35923

thus the eigenvalues of the original problem γ_i, $i \in \mathbb{N}$ are thought of as the limits of the zeroes of the corresponding truncated determinant γ_i^N. As is it the case in [3] (and references cited therein) the more N is increased, the larger the magnitude of the almost stable zeroes γ_i^N. Table 1 reveals some idea about this argument. Similarly if γ is an eigenvalue of the problem, the corresponding solution $\Phi_{n-}(\gamma)$, $n \in \mathbb{N}^+$ to within a multiplicative factor and that of the problem thereby can also be thought of as limiting case of the corresponding solution of the truncated system. To obtain these approximate unknowns $\Phi_{0-}(\gamma), \Phi_{1-}(\gamma), \ldots, \Phi_{N-}(\gamma)$, we can set one of them equal 1/2, for example $\Phi_0(\gamma)$. The other components $\Phi_{n-}(\gamma), n = 1, 2, \ldots, N$, can immediately be obtained by solving the reduced inhomogeneous system:

$$\Phi_{\ell-}(\gamma) + 2\sum_{n=1}^{N} \Gamma_n(\gamma) N_{n\ell} \Phi_{n-}(\gamma) = N_{0\ell}, \quad \ell = 1, 2, \ldots, N, \quad (2.25)$$

while the truncation of (2.24′) will be automatically satisfied. It is appropriate to recall that the above mentioned approximated solution exists in general even when N increases indefinitely since γ is a zero of the determinant of the truncated homogeneous system (2.24) and (2.24′) but not that of its corresponding inhomogeneous one.

Now, the solution $\varphi_-(x;\gamma)$ belongs to L_ρ space where [8]

$$1 < \rho < \frac{4}{3} \quad (2.26)$$

it follows that its Fourier representation is an ℓ_p-sequence where [9]

$$p = \frac{\rho}{\rho - 1} > 4. \quad (2.27)$$

Thus we have to investigate the justification of the truncation applied to systems with solutions in ℓ_p rather than the special case $p = 2$, that corresponds to the most familiar and widely used L_2-space. In the next section we achieve this study by closely following the study in [10] for the special case $p = \rho = 2$.

3. Infinite algebraic systems with ℓ_p-solutions

If the coefficients a_{jk} and b_j are subjected to the conditions

$$\sum_{j,k=1}^{\infty} |a_{jk}|^p < \infty, \quad \sum_{j=1}^{\infty} |b_j|^p < \infty, \quad (3.1)$$

it is required to find the solution ξ_k, $k \in \mathbb{N}$ where

$$\sum_{k=1}^{\infty} |\xi_k|^p < \infty, \quad (3.2)$$

of the infinite algebraic system

$$\xi_j - \lambda \sum_{k=1}^{\infty} a_{jk} \xi_k = b_j \quad (j \in \mathbb{N}), \quad (3.3)$$

as a limit of the solution of the truncated system

$$\xi_j - \lambda \sum_{k=1}^{N} a_{jk} \xi_k = b_j \quad (j = 1, 2, \ldots, N), \quad (3.4)$$

as $N \to \infty$. In this case the solution of system (3.4) can be considered as an approximation to that of system (3.3).

System (3.3) can be written in the form

$$Kx \equiv x - \lambda H x = y \quad (x = \{\xi_k\},\ y = \{b_k\}), \quad (3.5)$$

where H is a continuous linear compact operator, in $\ell^p = \chi$, defined by the matrix of the system

$$z = Hx \quad (x = \{\xi_k\}, \quad z = \{\eta_k\}). \quad (3.6)$$

We define a complete subspace $\bar{\chi} = \ell_N^p \subset \ell^p$ as the set of elements of $\ell^p = \chi$ whose coordinates are equal to zero starting from the $(N+1)^{\text{th}}$ coordinate. Moreover we denote the complete finite dimensional Euclidean space ℓ_N^p by $\bar{\chi}$. Thus, according to the Banach theorem, a linear continuous operator ϕ_0 mapping $\bar{\chi}$ in a one-to-one manner into $\bar{\chi}$ will has a continuous inverse operator ϕ_0^{-1}. The extension ϕ of the operator ϕ_0 on χ can be defined as $\phi = \phi_0 P$, where P is a projection operator, $P^2 = P$ mapping from χ into $\bar{\chi}$. The extension ϕ associates with $x = \{\xi_m\} \in \ell^p$ one element $\bar{x} = \phi x = (\xi_1, \xi_2, \ldots, \xi_n) \in \ell_n^p$.

Obviously

$$\|\phi\| = \|\phi_0\| = \|\phi_0^{-1}\| = 1. \quad (3.7)$$

Now, system (3.4) can be written in the form

$$\overline{K}\bar{x} = \bar{x} - \lambda \overline{H}\bar{x} = \phi y, \quad \overline{K} = \phi_0 \widetilde{K} \phi_0^{-1}, \quad \text{and}$$
$$\overline{H} = \phi_0 \widetilde{H} \phi_0^{-1}, \quad (3.8)$$

where the operator $\widetilde{H}$ is defined by the truncated matrix

$A_N = \{a_{ij}\}\ (i, j = 1, 2, \ldots, N),$

We use our notation to formulate a theorem [10] on the convergence of the sequence of approximate solutions as follows:

Theorem 1. *Let an operator $K = I + H$ has a continuous inverse operator and the following conditions are satisfied*

1. *For any $\bar{x} \in \bar{\chi}$, we have $\|\overline{H}\phi_0\bar{x} - \phi H\bar{x}\| \leqslant \beta\|x\|$,*
2. $\|Hx - \bar{x}\| \leqslant \beta_1\|x\|$,
3. *There exists $\bar{y} \in \bar{\chi}$ such that $\|y - \bar{y}\| \leqslant \beta_2\|y\|$,*
4. $\lim_{N\to\infty}\beta = \|\phi_0^{-1}\| = \lim_{N\to\infty}\beta_1 = \|\phi_0^{-1}\phi\| = \lim_{N\to\infty}\beta_2 = \|\phi_0^{-1}\phi\| = 0$

Then for sufficiently large N Eq. (3.4) is solvable and the convergence of the approximate solutions to the exact solution holds. In addition

$$\|x^* - \bar{x}_n^*\| \leqslant Q\beta\|\phi_0^{-1}\| + Q_1\beta_1\|\phi_0^{-1}\phi\| + Q_2\beta_2\|\phi_0^{-1}\phi\|,$$

where $\bar{x}_N^* = \phi_0^{-1}\bar{x}_n^*$, x^* is a solution of Eq. (3.5) and $\bar{x}^*$ is a solution of the equation

$$\widetilde{K}\bar{x} = \bar{x} - \lambda\widetilde{K}\bar{x} = Py, \quad \bar{x} = \phi_0^{-1}\bar{x},$$

$\widetilde{H}$ is a linear continuous operator in $\bar{\chi}$.

We now prove that conditions 1, 2, and 3 are satisfied. We have for any $\bar{x} = \phi x = (\xi_1, \xi_2, \ldots, \xi_N, 0, \ldots) \in \bar{\chi}$

$$\bar{z} = \phi H\bar{x} - \widetilde{H}\phi_0\bar{x}, \qquad \bar{z} = (\eta_1, \eta_2, \ldots \eta_N),$$
$$\eta_j = \sum_{k=1}^{N} a_{jk}\xi_k - \sum_{k=1}^{N} a_{jk}\xi_k = 0, \quad (j = 1, 2, \ldots, N).$$

Thus condition 1 is satisfied for $\beta = 0$.

To verify condition 2, we take any $x = \{\xi_n\} \in \ell^p$ and set $\bar{x} = [\phi x]_N = (\eta_1, \eta_2, \ldots, \eta_N)$, then

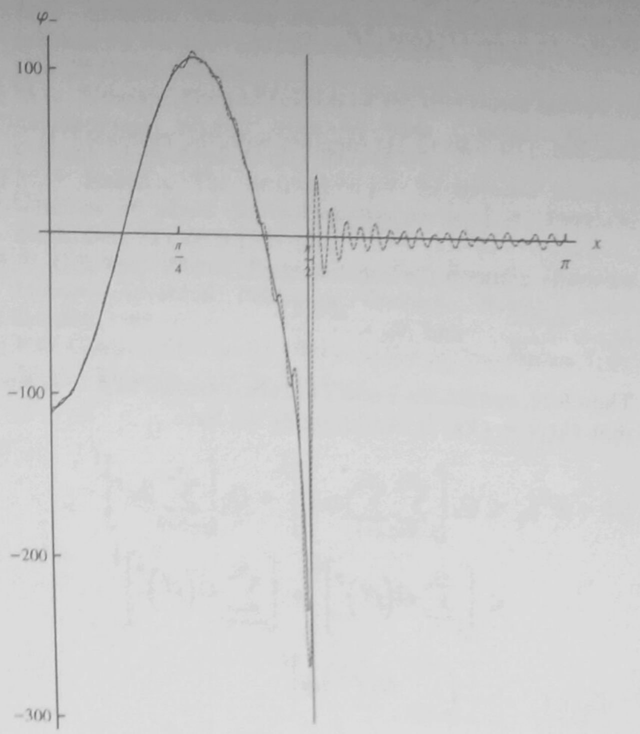

Fig. 1 The graph of the singular extension of the Neumann data $\tilde{\varphi}_-(x,\gamma_4)$ on Γ_{++}, the winding curve corresponds to the first 70 terms of its Fourier representation, while the continuous curve is obtained through its closed form in which the same number of Fourier components are considered.

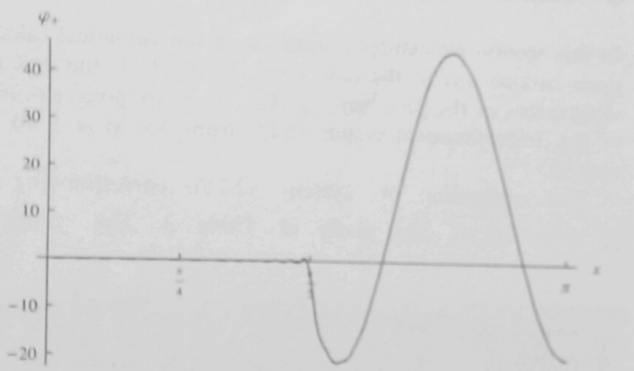

Fig. 2 The graph of the extension of the Hölder – continuous Dirichlit data $\varphi_+(x,\gamma_4)$ on Γ_- according to its Fourier representation in which the first 70 terms are considered.

$\Phi_{n-}\left(\gamma_4^{(26)}\right)$, $n > 26$ are approximately obtained through Eq. (2.25) in which $N = 26, \Phi_{n-}$ are substituted by means of their values and $\ell$is allowed to exceed 26. Some of the values thus obtained are given in the table between brackets. Note that they are almost the same which are directly obtained through solving system (2.25) at $N = 30$. In fact, according to relations (2.25), (2.11), and (3.12), the first neglected term in the expression of $\Phi_{(N+1)-}(\gamma)$ is

$$2\Gamma_{(N+1)}(\gamma)N_{(N+1),(N+1)}\Phi_{(N+1)-}(\gamma) < o(N^{-3}).$$

Finally the graphs of $\tilde{\varphi}_\mp(x,\gamma_4)$ are given in Figs. 1 and 2 respectively. In Fig. 1 the dotted curve of the singular function $\tilde{\varphi}_-(x,\gamma_4)$ corresponds to the Fourier representation

$$\tilde{\varphi}_-(x;\gamma_4) = 0.5 + 2\sum_{n=1}^{70}\Phi_{n-}(\gamma_4)\cos(nx).$$

It is clearly still so far from a stable form, in contrast to the curve of $\tilde{\varphi}_+(x,\gamma_4)$, Fig. 2 obtained through

$$\tilde{\varphi}_+(x;\gamma_4) = -0.5R_0(\gamma_4) - 2\sum_{n=1}^{70}R_n(\gamma_4)\Phi_{n-}(\gamma_4)\cos(nx),$$

moreover, this dotted curve conceals the singularity whatever be the order of the truncation. The coefficients $\Phi_{n-}(\gamma_4)$, $30 < n < 70$ are obtained by means of the above illustrated approximation.

The continuous curve in Fig. 1 obtained through the explicit formula (2.15) confirms the usefulness of the suggested modification since it surmounts both of these shortcomings.

5. Conclusion

It has been shown through a concrete example that the integral equation formulation of mixed Sturm–Liouville problems can be modified to complete the definition of partially prescribed conditions at one of the boundaries and concerned with quantities that inevitably may become infinite at the points where the boundary condition changes. This modification has the advantage of extending the domain of solution to the class of integrable functions and defining the singular solutions in a closed form. It is reached by reducing the problem to a homogenous Hilbert-type integral equation rather than the Cauchy-type one used in Ref. [1–4].

Since the closed form of the solution is defined in terms of its Fourier components, the problem is further reduced to an infinite homogenous system of algebraic equations and the completion of the solution is encountered with some difficulties. The justification of the truncation of a homogenous system (operator) involving an additional unknown constant, as in this case, may catastrophically influence the values of the Fourier components through affecting the zeroes of the determinant of the homogenous system truncated at any order. This constant which is inherent in the solution of a Hilbert-type integral equation could be defined in terms of the zero order component and the justification could be achieved using a similar idea to that followed in Refs. [3,4]. To this end, the well-known theorem concerning the study of the truncation of the infinite algebraic system of ℓ_2-solutions [10] is generalized to include the general case since the Fourier components of a singular integrable function constitute an ℓ_p-sequence.

Acknowledgment

The authors are indebted to Mr. M.T. Helal for his sincere help on carrying out the numerical calculations of this work.

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$$\|Hx - \bar{x}\|^p = \sum_{j=N+1}^{\infty}\sum_{k=1}^{\infty}|a_{jk}\xi_k|^p$$
$$\leqslant \sum_{j=N+1}^{\infty}\left|\left[\sum_{k=1}^{\infty}|a_{jk}|^q\right]^{1/q}\left[\sum_{k=1}^{\infty}|\xi_k|^p\right]^{1/p}\right|^p$$
$$= \sum_{j=N+1}^{\infty}\left|\left[\sum_{k=1}^{\infty}|a_{jk}|^q\right]^{p/q}\left[\sum_{k=1}^{\infty}|\xi_k|^p\right]\right|$$
$$= \sum_{j=N+1}^{\infty}\left[\sum_{k=1}^{\infty}|a_{jk}|^q\right]^{p/q}\|x\|^p.$$

Here, use has been made of Hölder inequality

$$\sum_{i=1}^{\infty}|\xi_i\eta_i| \leqslant \left[\sum_{i=1}^{\infty}|\xi_i|^p\right]^{1/p}\left[\sum_{i=1}^{\infty}|\eta_i|^q\right]^{1/q}, \quad \frac{1}{p}+\frac{1}{q}=1.$$

Thus

$$\|Hx - \bar{x}\| \leqslant \left[\sum_{j=N+1}^{\infty}\left[\sum_{k=1}^{\infty}|a_{jk}|^{\frac{p}{p-1}}\right]^{p-1}\right]^{1/p}\|x\| = \beta_1\|x\|.$$

It is clear that $\beta_1 \to 0$ as $N \to \infty$. Next putting $\bar{y} = [y]_N$, we have

$$\|y - \bar{y}\| = \left[\sum_{j=N+1}^{\infty}|b_j|^p\right]^{1/p} = \frac{\left[\sum_{j=N+1}^{\infty}|b_j|^p\right]^{1/p}}{\|y\|}\|y\| = \beta_2\|y\|$$

Again $\beta_2 \to 0$ as $N \to \infty$. finally, condition 4 is obviously fulfilled. Thus the following theorem has been established.

Theorem 2. *Suppose that:*

1. *The homogeneous system corresponding to system (3.5) has only trivial solution in* ℓ^p,
2. $\sum_{j=1}^{\infty}\left[\sum_{k=1}^{\infty}|a_{jk}|^{\frac{p}{p-1}}\right]^{p-1} < \infty$,
3. $\sum_{j=1}^{\infty}|b_j|^p < \infty$,

then the infinite system (3.3) has a unique solution in ℓ^p. *The truncated system will also have a unique solution and the following estimate holds*

$$\|\chi - \chi^N\|_{\ell_p} \leqslant Q_1\left[\sum_{j=N+1}^{\infty}\left[\sum_{k=1}^{\infty}|a_{jk}|^{\frac{p}{p-1}}\right]^{p-1}\right]^{\frac{1}{p}} + Q_2\left[\frac{\sum_{j=N+1}^{\infty}|b_j|^p}{\sum_{j=1}^{\infty}|b_j|^p}\right]^{\frac{1}{p}}, \tag{3.9}$$

where Q_1 *and* Q_2 *are constants.*

In the case of the infinite system corresponding to system (2.24) we have

$$a_{jk} = \Gamma_k N_{kj}, \quad \text{and } b_j \to N_{0j},$$

and we have already indicated that the homogenous system corresponding to it may not has a nontrivial solution if γ is an eigenvalue of the problem. Additionally, taking into account formula (2.20) for $k = j$,

$$N_{kj} = -\frac{1}{2}\frac{k+1}{k-j}[P_k(\cos c)P_{j+1}(\cos c) - P_{k+1}(\cos c)P_j(\cos c)] \text{ for } k \neq j \tag{3.10}$$

(see Ref. [7]), and (2.21) together with the estimate [11]

$$|P_k(\cos c)| \leqslant \left(\frac{2}{\pi}\right)^{1/2}\frac{1}{\sqrt{k\sin c}} \tag{3.11}$$

we easily come to the conclusions

$$N_{n\ell} \sim \frac{\text{const.}}{\sqrt{n\ell^3}}, \quad \text{and } N_{0\ell} \sim \frac{\text{const.}}{\sqrt{\ell}}. \tag{3.12}$$

Therefore, conditions 2 and 3 are also satisfied as $p > 4$. Recall that $\Gamma_k(\gamma) = O(k^{-2})$. Additionally we have

$$\|\Phi - \Phi^N\|_{\ell_p} \leqslant Q_1\left[\sum_{j=N+1}^{\infty}\sum_{k=1}^{\infty}|a_{jk}|^p\right]^{\frac{1}{p}} + Q_2\left[\sum_{j=N+1}^{\infty}|b_j|^p\right]^{\frac{1}{p}}$$
$$\leqslant \left[\sum_{j=N+1}^{\infty}O\left(j^{\frac{-3}{2}}\right)^p\right]^{\frac{1}{p}} + \left[\sum_{j=N+1}^{\infty}O\left(j^{\frac{-1}{2}}\right)^p\right]^{\frac{1}{p}}$$
$$\leqslant \left[\int_{j=N+1}^{\infty}O\left(j^{\frac{-3p}{2}}\right)dj\right]^{\frac{1}{p}}$$
$$+ \left[\int_{j=N+1}^{\infty}O\left(j^{\frac{-p}{2}}\right)dj\right]^{\frac{1}{p}}$$
$$\leqslant \frac{c_1}{(N+1)^{\frac{3p-2}{2p}}} + \frac{c_2}{(N+1)^{\frac{p-2}{2p}}} \leqslant \frac{c_3}{(N+1)^{\frac{p-2}{2p}}}, \tag{3.13}$$

where c_1, c_2, and c_3 are constants

4. Numerical example

In this section we exhibit the results of the numerical calculations carried out in the case $c = \frac{\pi}{2}$. In Table 1, the first five eigenvalues of the problem γ_i^N, $i, 1, 2, \ldots, 5$, are given as zeroes of the determinant of system (2.25) truncated at $N = 10$, 20, and 30.

The solutions of system (2.25) corresponding to $\gamma_4^{(N)}$, $N = 26, 30$, are given in Table 2. The values of

Table 2 The Fourier coefficients of $\varphi_-(x, \gamma_4^N)$, $N = 26$ and $N = 30$.

$\Phi_{n-}\left(\gamma_4^{(26)}\right)$	1	3.370	−7.792	−41.48	−56.79	−24.35
$\Phi_{n-}\left(\gamma_4^{(30)}\right)$	1	3.371	−7.795	−41.49	−56.81	−24.36
$\Phi_{n-}\left(\gamma_4^{(26)}\right)$	15.63	13.57	−11.06	−10.45	9.124	8.870
$\Phi_{n-}\left(\gamma_4^{(30)}\right)$	15.64	13.58	−11.07	−10.46	9.144	8.876
$\Phi_{n-}\left(\gamma_4^{(26)}\right)$	−8.022	−7.871	7.262	7.164	−6.699	−6.628
$\Phi_{n-}\left(\gamma_4^{(30)}\right)$	−8.023	−7.877	7.262	7.170	−6.699	−6.634
$\Phi_{n-}\left(\gamma_4^{(26)}\right)$	6.259	6.201	−5.903	−5.850	5.606	5.554
$\Phi_{n-}\left(\gamma_4^{(30)}\right)$	6.258	6.208	−5.900	−5.857	5.600	5.564
$\Phi_{n-}\left(\gamma_4^{(26)}\right)$	−5.355	−5.296	5.155	(5.097	−4.963	−4.925)
$\Phi_{n}\left(\gamma_4^{(30)}\right)$	−5.345	−5.310	5.125	5.088	−4.934	−4.891

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