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ON THE SOLUTIONS OF MIXED PLANE PROBLEMS THAT ARE UNBOUNDED WHERE THE BOUNDARY CONDITIONS CHANGE

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Abstract. In this paper, we apply a modification to the method of the integral equation formulation of mixed plane boundary value problems so that it enables us to obtain the solutions unbounded at the points where the boundary conditions change. Such solutions are of great physical interest. The modification is illustrated by means of a typical problem. As is it the case in the original method proposed by Cherskii [1], the problem is reduced to an infinite system of algebraic equations. The justification of the truncation of such systems has been established.

1. Introduction

In 1961, Cherskii proposed the method of the discrete Riemann problems for solving stationary finite plane mixed problems [1]. This method, originally proposed for solving stationary Dirichlet-Neumann problems, consists of reducing the problem to an integral equation. The unknown of this integral equation is an extension of one of the partially imposed conditions and is compatible to the other. It has become of wide applications in several branches of mathematical physics [2]. Further, El-Sheikh could modify it to solve initial Dirichlet-Neumann problems [3] (with Eckhardt) and [4]. He also performed another modification to the method for solving steady Dirichlet-Newton problems [5] (with Gad-Allah) as well as initial Dirichlet-Newton problems [6]. Additionally, the method has been applied to time dependent elastic contact problems [7]. In all these works, however, it is the extensions of physically continuous entities that was determined by means of the procedures followed in [1-7]. On the other hand, solutions involving singularities at the points where the boundary conditions change are of great physical interest. For example, in the problems of heat conductivity, the Dirichlet Condition represent the temperature which is always continuous, but if this condition changes to a Neumann one at some point on the boundary, the heat flow the later represents, is in general discontinuous with probably an infinite jump there. Another example is usually met in the contact problems of the theory of elasticity. The solution of such problems requires the determination of both the normal displacement as well as the normal stress on the arc of contact. Again, the displacement is bounded every where while the normal stress may be unbounded at the end points of such arcs. It is this unboundedness

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which is responsible for the wear. Physically, it is impossible to subject one point to more than one condition and mathematically it is known that discontinuities and/or singularities in the solutions of mixed problems are inevitably presented at points where the boundary conditions change [8]. At this point, the determination of the solution(s) involving singularities through the procedures followed in [1-7] is doubted. As already pointed out above, in these works, the unknown functions in the integral equation to which the problem was reduced are necessarily continuous. However, the solution of such integral equations and/or the very form of its corresponding discrete problem lead to make assumption(s) about such unknown function. Thus other solutions which do not satisfy these assumptions could have been exist and the solution(s) involving singularities for quantities other than these unknown functions may indeed be lost. In particular, the physically important unbounded solutions may indeed be thrown out. For example [3], the discrete problem to which the mixed Dirichlet-Neumann Sturm-Liuoville problem for the reduced wave equation in the circle is

$$\Phi_{n+}(\gamma) = |n|\Phi_{n-}(\gamma) + Q_{|n|}(\gamma)\Phi_{n-}(\gamma), \qquad (n \in Z). \tag{1.1}$$

Here $\Phi_{n-}(\gamma)$ and $\Phi_{n+}(\gamma)$ represent the Fourier components of the temperature φ_- and the heat flow φ_+ on the circumference respectively, γ is the parameter of separation of variables and $Q_{|n|}(\gamma)$ are well-defined coefficients. From this equation, it is clear that the Fourier representation $\{\Phi_{n+}(\gamma)\}$ of the heat flow is possible only if the temperature function φ_- is a Hoelder continuous one. Further, a discrete problem like (1.1) expresses an unbounded solution for φ_+, if any, as an infinite series and the singularity is practically lost in this way. It is then necessary to rearrange the discrete problem such that the unknown function in its corresponding integral equation becomes the function that can increase indefinitely and then search for the unbounded solution. To this end, the procedures in [1-7] ceases to be applicable and modifications should be achieved.

In this work, the required modification is outlined through consideration of a typical example. Namely, the propagation of harmonic heat waves in a cylinder due to two symmetric arc-sources of the form $e^{-i\omega t}f(\theta)$ while isolated every where else on the boundary. This constitutes simultaneously the thermal part of the thermoelastic contact problem of symmetric indentation of two punches in the form of circular segments into the exterior surface of the cylinder. In a way similar to that followed in [1-7], this problem is reduced to a discrete Riemann problem such as (1.1) but in which φ_+ represents the continuous temperature. Thus the problem is converted to a singular integral equation with Hilbert kernel (rather than a Cauchy's one) in which the unknown function is the heat flow. The unbounded solution of this equation can simply be obtained and it can be further reduced to an infinite system of algebraic equations in which the unknowns are its Fourier coefficients. These coefficients are found in the space $l_p (p > 4)$. The error occuring due to the truncation has been estimated.

2. The Problem and Its Integral Equation Formulation

The point of departure is the following heat equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{1 \partial T}{r \partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = \frac{1}{a_T}\frac{\partial T}{\partial t}, \tag{2.1}$$

with the mixed boundary conditions

$$T(1,\theta;t) = e^{-i\omega t} f(\theta), \quad \text{if } \theta \in \Gamma_1, \tag{2.2}$$

$$\frac{\partial T(1,\theta;t)}{\partial r} = 0, \quad \text{if } \theta \in \Gamma_2, \tag{2.3}$$

$$|T(r,\theta;t)| < \infty \tag{2.4}$$

where $\Gamma_1 = [-\alpha,\alpha] \bigcup [\pi-\alpha,\pi+\alpha]$, $\Gamma_2 = [-\pi,\pi] \setminus \Gamma_1$, a_T is the thermal diffusivity and $f(\theta)$ is an even function with respect to θ.

The mixed boundary conditions (2.2) and (2.3) can be replaced by the two uniform and compatible ones:

$$u(1,\theta) = f_-(\theta) + \varphi_+(\theta), \qquad \frac{\partial u(1,\theta)}{\partial r} = \varphi_-(\theta) \tag{2.5}$$

where

$$T(r,\theta;t) = e^{-i\omega t} u(r,\theta),$$

$$\varphi_+(\theta) = \begin{cases} 0 & \text{if } \theta \in \Gamma_1, \\ \text{undetermined} & \text{if } \theta \in \Gamma_2, \end{cases}$$

$$\varphi_-(\theta) = \begin{cases} \text{undetermined} & \text{if } \theta \in \Gamma_1, \\ 0 & \text{if } \theta \in \Gamma_2, \end{cases}$$

$$f_-(\theta) = \begin{cases} \text{f}(\theta) & \text{if } \theta \in \Gamma_1, \\ 0 & \text{if } \theta \in \Gamma_2. \end{cases} \tag{2.6}$$

From the definition of u combined with equation (2.1) we get

$$\Delta u + \frac{i\omega}{a_T} u = 0.$$

Applying the finite Fourier transforms with respect to θ to this equation, the solution of the resulting equation that satisfies (2.4) is

$$U_n(r) = A_n J_{|n|}(\gamma r), \qquad (n \in Z),$$

where

$$\gamma^2 = \frac{i\omega}{a_T}\ , \ U_n(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(r,\theta) e^{-in\theta} d\theta$$

and $J_{|n|}$ are Bessel functions of the first kind. Substituting this solution in the Fourier transforms of the boundary conditions (2.5) and eliminating A_n, we arrive at the following discrete Riemann problem

$$\Phi_{n+} = \frac{J_{|n|}(\gamma)}{J'_{|n|}(\gamma)} \Phi_{n-} - F_{n-}, \qquad (n \in Z), \tag{2.7}$$

where

$$\Phi_{n\pm} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_{\pm}(\theta)e^{-in\theta}d\theta \text{ and } F_{n-} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_{-}(\theta)e^{-in\theta}d\theta.$$

Multiplying (2.7) by n, it can be rewritten in the form

$$n\Phi_{n+} = sgn(n+\frac{1}{2})\Phi_{n-} - \Gamma_n\Phi_{n-} - nF_{n-}, \qquad (n \in Z - \{0\}), \tag{2.8}$$

where $\Gamma_n = sgn(n+\frac{1}{2}) - \frac{nJ_{|n|}(\gamma)}{J'_{|n|}(\gamma)}$ and $|\Gamma_n| = 0\left(\frac{1}{n^2}\right)$.

On multiplying (2.7) by n the first equation, involving Φ_{0-}, was lost. An alternative equation restoring the hypothesis necessary for determining the solution of (2.7) can be obtained from the first of conditions (2.5) at $\theta = 0$ where it always holds for any $\alpha \in \left(0, \frac{\pi}{2}\right]$, this yields

$$-\frac{J_0(\gamma)}{\gamma J_1(\gamma)}\Phi_{0-} + 2\sum_{n=1}^{\infty}\frac{J_n(\gamma)}{J'_n(\gamma)}\Phi_{n-} = f(0). \tag{2.9}$$

Performing the inverse Fourier transform

$$W^{-1}\Phi_{n\pm} = \sum_{n=-\infty}^{\infty}\Phi_{n\pm}e^{in\theta} = \varphi_{\pm}(\theta) \tag{2.10}$$

and using the formula

$$W^{-1}sgn(n+\frac{1}{2})\Phi_{n-} = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\varphi_{-}(t)e^{it}}{e^{it}-e^{i\theta}}dt \tag{2.11}$$

which can be found in [2] (taking into considerations that $\varphi'_{+}(x) = 0$ as $\theta \in \Gamma_1$) we reduce the discrete problem (2.8) to the singular integral equation

$$\frac{1}{\pi}\left(\int_{-\alpha}^{\alpha}\frac{\varphi_{-}(t)}{1-e^{i(\theta-t)}}dt + \int_{\pi-\alpha}^{\pi+\alpha}\frac{\varphi_{-}(t_1)}{1-e^{i(\theta-t_1)}}dt_1\right) = \sum_{n=-\infty}^{\infty}\Gamma_n\Phi_{n-}e^{in\theta} - if'_{-}(\theta).$$

Taking into account that $\varphi_{-}(\theta)$ is a π-periodic function the substitution $t_1 = t+\pi$ leads to the result

$$\int_{\pi-\alpha}^{\pi+\alpha}\frac{\varphi_{-}(t_1)dt_1}{1-e^{i(\theta-t_1)}} = \int_{-\alpha}^{\alpha}\frac{\varphi_{-}(t)dt}{1+e^{i(\theta-t)}}$$

and the above equation assumes the form

$$\frac{2}{\pi}\int_{-\alpha}^{\alpha}\frac{\varphi_{-}(t)e^{2it}}{e^{2it}-e^{2i\theta}}dt = \sum_{n=-\infty}^{\infty}\Gamma_n\Phi_{n-}e^{in\theta} - if'(\theta).$$

Making use of the formula

$$\frac{2e^{2it}}{e^{2it}-e^{2i\theta}} = 1 - i\cot(t-\theta)$$

as well as the odd property of Γ_n the above equation assumes the form

$$\frac{1}{\pi}\int_{-\alpha}^{\alpha} \cot(t-\theta)\varphi_-(t)dt = -2\sum_{n=1}^{\infty}\Gamma_n\Phi_{n-}\sin n\theta + f'(\theta), \theta \in (-\alpha, a). \tag{2.12}$$

The final form of eq. (2.12) can be obtained by taking into consideration that the functions $\varphi_\pm(\theta)$ and $f_-(\theta)$ are symmetric about $\theta = \frac{\pi}{2}$. Thus, expressing, for example, the relation $\varphi_-\left(\frac{\pi}{2}+y\right) = \varphi_-\left(\frac{\pi}{2}-y\right)$ in terms of its Fourier expansion

$$\varphi_-(\theta) = \sum_{n=-\infty}^{\infty}\Phi_{n-}e^{in\theta}$$

with one eye on the definition of Φ_{n-}, we simply get the result

$$\Phi_{n-} = \left.\begin{matrix}\frac{1}{\pi}\int_{-\alpha}^{\alpha}\varphi_-(\theta)e^{-in\theta}d\theta \text{ if } n \text{ is even} \\ 0 \qquad\qquad \text{if } n \text{ is odd}\end{matrix}\right\} \tag{2.13}$$

and equation (2.12) becomes

$$\frac{1}{\pi}\int_{-\alpha}^{\alpha} \cot(t-\theta)\varphi_-(t)dt = f'(\theta) - 2\sum_{n=1}^{\infty}\Gamma_{2n}\Phi_{2n-}\sin 2n\theta, \theta \in (-\alpha, \alpha). \tag{2.14}$$

3. The Reduction to Algebraic System

The Hilbert-type integral equation (2.14) can be inverted in the class of integrable functions [9], with the result.

$$\varphi_-(\theta) = \frac{1}{X(\theta)}\left[a_0\cos\theta + 2\sum_{n=1}^{\infty}\Gamma_{2n}\Phi_{2n-}V_n(\theta) - m(\theta)\right] \tag{3.1}$$

where a_0 is a constant,

$$X(\theta) = \sqrt{2(\cos 2\theta - \cos 2\alpha)}, \qquad m(\theta) = \frac{1}{\pi}\int_{-\alpha}^{\alpha}\frac{X(t)f'(t)dt}{\sin(t-\theta)}$$

and

$$V_n(\theta) = \frac{1}{\pi}\int_{-\alpha}^{\alpha}\frac{X(t)\sin 2nt dt}{\sin(t-\theta)}. \tag{3.2}$$

The explicit expressions of the integrals $V_n(\theta)$ are

$$V_n(\theta) = \sum_{m=0}^{n}\mu_{n-m}(\cos 2\alpha)\cos(2m+1)\theta$$

where

$$\mu_k(\cos\alpha) = \frac{P_{k-2}(\cos\alpha) - P_k(\cos\alpha)}{2k-1} \qquad (k = 2, 3, \cdots)$$

$$\mu_0(\cos\alpha) = 1,\ \mu_1(\cos\alpha) = -\cos\alpha \tag{3.3}$$

and $P_n(\cos\alpha)$ is a Legender polynomial defined by the formula

$$P_n(\cos\alpha) = \frac{1}{\pi}\int_{-a}^{a}\frac{\cos(n+\frac{1}{2})xdx}{X(x)}. \tag{3.4}$$

Note that $m(\theta)$ can be expressed as a superposition of $V_n(\theta)$. In order to define the coefficients Φ_{2n-} in eq. (3.1) we substitute instead of $\varphi_-(x)$ its expression according to this equation in (2.13). Thus we arrive at the following infinite system of liner algebraic equations

$$\Phi_{2\ell-} = a_0 R_\ell + 2\sum_{n=1}^{\infty}\Gamma_{2n}N_{n\ell}\Phi_{2n-} - M_\ell \qquad (\ell\in N^+) \tag{3.5}$$

where

$$N_{n\ell} = \frac{1}{\pi}\int_{-\alpha}^{\alpha}\frac{V_n(\theta)\cos 2\ell\theta}{X(\theta)}d\theta,\quad R_\ell = \frac{1}{\pi}\int_{-\alpha}^{\alpha}\frac{\cos 2\ell\theta\cos\theta d\theta}{X(\theta)}$$

and

$$M_\ell = \frac{1}{\pi}\int_{-\alpha}^{\alpha}\frac{m(\theta)\cos 2\ell\theta}{X(\theta)}d\theta. \tag{3.6}$$

In view of (3.3) and (3.4) these coefficients can simply be found:

$$N_{n\ell} = \frac{1}{4}\sum_{m=0}^{n}\mu_{n-m}(\cos 2\alpha)\left[P_{m-\ell}(\cos 2\alpha) + P_{m+\ell}(\cos 2\alpha)\right]$$

and

$$R_\ell = \frac{1}{4}\left[P_\ell(\cos 2\alpha) + P_{\ell-1}(\cos 2\alpha)\right]. \tag{3.7}$$

Again M_ℓ can be expressed as a composition of $N_{n\ell}$, $n = 1, 2, \cdots$.

4. The Truncation of the Algebraic System

Since system (3.5) can in general be solved only approximately, namely using the method of truncation, we set up function spaces and sequence spaces. The solution (3.1) of equation (2.14) is an $L_\rho[-a,a]$, were $1 < \rho < \frac{4}{3}$[10]. Consequently the Fourier coefficients Φ_{2n-} will belong to ℓ_p, where $p = \frac{\rho}{\rho-1}$[11]. Thus we will work in the space $\ell_p(p>4)$ with the norm

$$\|\Phi\|_{\ell_p} = \left(\sum_{n=0}^{\infty}|\Phi_{n-}|^p\right)^{\frac{1}{p}} \tag{4.1}$$

where $\Phi = \{\Phi_{n-}\}_{n=\overline{0,\infty}}$. The justification of truncating system (3.5) is a simple consequence of the following theorem whose proof is similar to that given in [12] for the case $p = 2$.

Theorem. *Suppose that*

(i) *The homogeneous system corresponding to system (3.5) has only trivial solution in* ℓ_p

(ii)
$$\sum_{\ell=0}^{\infty}\Big(\sum_{n=1}^{\infty}|\Gamma_{2n}N_{n\ell}|^{\frac{p}{p-1}}\Big)^{p-1}<\infty,$$

(iii)
$$\sum_{\ell=0}^{\infty}|R_\ell|^p<\infty,$$

then the infinite system (3.5) has a unique solution in ℓ_p. *The truncated system will also have a unique solution and the following estimate holds*

$$\|\Phi-\Phi^N\|_{\ell_p}\le Q_1\Big[\sum_{\ell=N+1}^{\infty}\Big(\sum_{n=1}^{\infty}|\Gamma_{2n}N_{n\ell}|^{\frac{p}{p-1}}\Big)^{p-1}\Big]^{\frac{1}{p}}+Q_2\left[\frac{\sum_{\ell=N+1}^{\infty}|R_\ell|^p}{\sum_{\ell=0}^{\infty}|R_\ell|^p}\right]^{\frac{1}{p}} \tag{4.2}$$

where Q_1 *and* Q_2 *are constants.*

We shall assume that the frequency ω differs from those values for which the homogeneous system corresponding to (3.5) has nontrivial solutions. The fulfillment of the second and the third conditions follows from (3.7) and (3.3) for $n=\ell$ together with the formula

$$N_{n\ell}=-\frac{1}{2}\frac{n+1}{n-\ell}\left[P_n(\cos 2\alpha)P_{\ell+1}(\cos 2\alpha)\;-\;P_{n+1}(\cos 2\alpha)P_\ell(\cos 2\alpha)\right];\quad n\ge 1,\ n\ne\ell \tag{4.3}$$

together with the estimate (formula 22.14.9 of [13])

$$|P_n(\cos a)|\le\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{1}{\sqrt{n}\sin a};\,(0<a<\pi,\quad n=1,2,\cdots). \tag{4.4}$$

Thus, we have

$$|N_{n\ell}|\sim\frac{c}{\sqrt{n\ell^3}}\quad\text{and}\quad R_\ell\sim\frac{c}{\sqrt{\ell}}. \tag{4.5}$$

Therefore, conditions (ii) and (iii) are satisfied as $p>4$. Recall that $\Gamma_k=0\left(k^{-2}\right)$. Additionally, we have

$$\|\Phi-\Phi^N\|_{\ell_p}\le\frac{c}{(N+1)^{\frac{p-2}{2p}}}. \tag{4.6}$$

Formula (3.1) defines $\varphi_-(\theta)$ only to within an arbitrary additive factor, $a\cos\theta$. This might be expected since it follows entirely from knowledges concerning the derivative $\varphi'_-(\theta)$ rather than the function itself. In eq. (2.8), $n\Phi_{n\pm}$ are proportional to the Fourier components of $\varphi'_\pm(\theta)$. In return, the solution of system (3.5) provides only the derivative of the required solution or equivelently the non zero order Fourier coefficients. The coefficient Φ_{0-} is that satisfies condition (2.9) together with the other coefficients determined through system (3.5). Thus, the whole solution can practicaly be obtained by replacing condition (2.9), of the selfconsistency, instead of the sole equation involving Φ_{0-} in system (3.5), the first one. Analogous way was used in [7] (section 6, *second way*). In this way, the solution of the N^{th}-order truncation of (2.9) together with the equations corresponding to $n = 1, 2, \cdots, N+1$ provides the coefficients $\Phi_{2n}, n = 0, 1, \cdots, N$ as well as the constant a all together with a precision that becomes infinite as $N \longrightarrow \infty$. The approximate solution of problem (2.1)-(2.4) is easily shown to be

$$T(r, \theta; t) = e^{-i\omega t}\left[\frac{\Phi_{0-}}{J_0'(\gamma)} J_0(\gamma r) + 2\sum_{n=1}^{N} \frac{\Phi_{2n-}}{J_{2n}'(\gamma)} J_{2n}(\gamma r)\cos 2n\theta\right] \tag{4.7}$$

where

$$J_n'(\gamma) = J_n'(\gamma r)|_{r=1}. \tag{4.8}$$

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