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The evolution of the motions of a rigid body close to the Lagrange case under the action of an unsteady torque☆

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ABSTRACT

Perturbed rotational motions of a rigid body, close to the Lagrange case, under the action of a torque that is slowly varying in time are investigated. Conditions for the possibility of averaging the equations of motion with respect to the nutation phase angle are presented and an averaged system of equations is obtained. An example, corresponding to the motion of a body in a medium with linear dissipation, is considered.

The problem of the evolution of the rotations of a rigid body about a fixed point continues to attract attention related to problems of the entry of aircraft into the atmosphere, cosmonautics, gyroscopes and the dynamics of a rotating missile. Here, in many cases, a motion in the Lagrange case can be considered as the generating motion of the body that takes account of the main torques acting on it. In this case, the body has a fixed point and is in a gravitational force field, and, moreover, the centre of mass of the body and the fixed point lie on the axis of dynamic symmetry of the body. A restoring torque similar to the gravitational torque, is created by the aerodynamic forces acting on the body in a gas stream. Motions close to the Lagrange case have therefore been investigated in a number of papers on aerodynamics where the restoring torque and different perturbing torques were taken into account.

We mention the analysis of the motion about the centre of mass of aircraft entering the atmosphere at a high velocity.[1,2] The motion of a rotating rigid body in the atmosphere under the action of a sinusoidal or biharmonic time-dependent restoring torque and small perturbing torques has been investigated.[3] A procedure for averaging over an Euler–Poinsot motion for a satellite with an arbitrary triaxial ellipsoid of inertia was constructed for the first time.[4] The perturbed motions of a rigid body, close to the Lagrange case, have been considered in a number of papers such as Refs 5–14, for example. A review of the results obtained up to 1998 on the problem of the evolution of the rotations of a rigid body, close to the Lagrange case, is available.[10] An averaging procedure for the slow variables in the first approximation of the perturbed motion of a rigid body, close to the Lagrange case, has been described and the perturbed rotational motions of a rigid body, close to a regular precession in the Lagrange case, have been studied for different orders of smallness of the projections of the angular momentum vector.[5,6,14]

The motion of a heavy symmetric rigid body with a fixed point under the action of friction forces due the surrounding dissipative medium has been considered.[7] The asymptotic behaviour of the motions of a Lagrange gyroscope, close to regular precessions, under the action of a small perturbing torque has been investigated.[8,9,12] The motion of a Lagrange gyroscope with a vibrating suspension has been studied [11] and the effect of fast periodic and conditionally periodic vibrations of the point of suspension on the existence and stability of the steady rotations of a Lagrange gyroscope about the vertical and its regular precessions [13] have been studied.

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1. Statement of the problem and unperturbed motion

The motion of a dynamically symmetric rigid body about a fixed point O in the case of perturbations of an arbitrary physical nature is considered. The equations of motion (the dynamic and kinematic Euler equations) have the form

$$\begin{aligned}
&A\dot{p} + (C - A)\,qr = \mu\sin\theta\cos\varphi + \varepsilon M_1 \\
&A\dot{q} + (A - C)\,pr = -\mu\sin\theta\sin\varphi + \varepsilon M_2 \\
&C\dot{r} = \varepsilon M_3; \quad M_i = M_i(p,q,r,\psi,\theta,\varphi,\tau), \quad i = 1,2,3 \\
&\dot{\psi} = (p\sin\varphi + q\cos\varphi)\operatorname{cosec}\theta, \quad \tau = \varepsilon t \\
&\dot{\theta} = p\cos\varphi - q\sin\varphi, \quad \dot{\varphi} = r - (p\sin\varphi + q\cos\varphi)\operatorname{ctg}\theta
\end{aligned} \tag{1.1}$$

Dynamic equations (1.1) are written in projections onto the principal axes of inertia of the body passing through the point O. Here, p, q, and r are the projections of the angular velocity vector onto these axes, M_i (i = 1, 2, 3) are the projections of the perturbing moment vector onto the same axes, and they depend on the slow time $\tau = \varepsilon t$, where $\varepsilon \ll 1$ is a small parameter characterizing the magnitude of the perturbations and t is the time), ψ, θ and φ are Euler angles, and A is the equatorial and C is the axial moment's of inertia about the point O $A \neq C$. It is assumed that a restoring torque acts on the body, the maximum magnitude of which is equal to μ and is created by a force, constant in magnitude and direction, that is applied to a certain fixed point of the axis of dynamic symmetry. In the case of a heavy top, we have $\mu = mgl$, where m is the body mass, g is the acceleration due to gravity and l is the distance from the fixed point O to the centre of gravity of the body.

We set the problem of investigating the asymptotic behaviour of the solutions of system (1.1) for a small ε; the analysis will be carried out by the method of averaging[15] on a time interval of the order of ε^{-1}.

In the case of the unperturbed motion, the quantities[14,16]

$$\begin{aligned}
&G_z = A\sin\theta\left(p\sin\varphi + q\cos\varphi\right) + Cr\cos\theta = c_1 \\
&H = \frac{1}{2}[A(p^2 + q^2) + Cr^2] + \mu\cos\theta = c_2, \quad r = c_3
\end{aligned} \tag{1.2}$$

are the first integrals of the equations for system (1.1) when $\varepsilon = 0$. Here, G_z is the projection of the angular momentum vector onto the vertical Oz, H is the total energy of the body, r is the projection of the angular velocity vector onto the axis of dynamic symmetry, c_i (i = 1, 2, 3) are arbitrary constants and $c_2 \geq -\mu$.

In the general case, the expression for the angle of nutation θ in the unperturbed motion is known as functions of the time t, the integrals of the motion (1.2) and the arbitrary phase constant β:[14,16]

$$\begin{aligned}
&u = \cos\theta = u_1 + (u_2 - u_1)\operatorname{sn}^2(\alpha t + \beta), \quad \operatorname{sn}(\alpha t + \beta) = \sin\operatorname{am}(\alpha t + \beta, k) \\
&\alpha = \left[\mu(u_3 - u_1)/(2A)\right]^{1/2}, \quad k^2 = (u_2 - u_1)(u_3 - u_1)^{-1}, \quad 0 \leq k^2 \leq 1
\end{aligned} \tag{1.3}$$

Here, u is the periodic function $\alpha t + \beta$ with a period $K(k)/\alpha$, sn and am are an elliptic sine and amplitude,[17] k is the modulus of the elliptic functions, and the real roots of the cubic polynomial

$$Q(u) = A^{-2}[(2H - Cr^2 - 2\mu u)(1 - u^2)A - (G_z - Cru)^2] \tag{1.4}$$

are denoted by u_1, u_2 and u_3.

The relations between its roots and first integrals (1.2) are written in the following way:

$$\begin{aligned}
&u_1 + u_2 + u_3 = \frac{H}{\mu} - \frac{Cr^2}{2\mu} + \frac{C^2r^2}{2A\mu}, \quad u_1u_2 + u_1u_3 + u_2u_3 = \frac{G_zCr}{A\mu} - 1 \\
&u_1u_2u_3 = -\frac{H}{\mu} + \frac{Cr^2}{2\mu} + \frac{G_z^2}{2A\mu} \\
&-1 \leq u_1 \leq u_2 \leq 1 \leq u_3 < +\infty
\end{aligned} \tag{1.5}$$

Formulae (1.2), (1.3) and (1.5) describe the solution of system (1.1) when $\varepsilon = 0$ in the Lagrange case.

2. Averaging procedure

An averaging procedure developed earlier[5,14] is used later for averaging system (1.1) in the case of perturbations depending on the slow time τ and allowing averaging over the phase of the nutation angle θ along the trajectories of change $\theta(t)$. We separate the fast and slow variables, and, here, first integrals (1.2) for the perturbed motion (1.1) are the slow variables. The fast variables are the angles of proper rotation φ, nutation 2 and precession ψ.

Using a number of transformations, we reduce the first three equations of (1.1) to the form[5,14]

$$\begin{aligned}
&\dot{G}_z = \varepsilon\left[(M_1\sin\varphi + M_2\cos\varphi)\sin\theta + M_3\cos\theta\right] \\
&\dot{H} = \varepsilon\left(M_1p + M_2q + M_3r\right) \\
&\dot{r} = \varepsilon C^{-1}M_3; \quad M_i = M_i\left(p,q,r,\psi,\theta,\varphi,\tau\right), \quad i = 1,2,3
\end{aligned} \tag{2.1}$$

Here, and in the last three kinematic equations of (1.1), it is implied that the variables p, q and r are expressed using relations (1.2) as functions of G_z, H, r, ψ, θ, φ which are substituted into Eqs (1.1) and (2.1).

However, the right-hand sides of Eqs (2.1) contain the three fast variables θ, φ and ψ which are periodic in t that makes it difficult to use the averaging method (the resonance problem). To eliminate this difficulty, it is required that the expressions on the right-hand sides of (2.1) are represented as functions of the slow variables and the nutation angle θ that are 2π-periodic in the phase of the angle θ and have the following structural properties of a perturbing torque (see equalities (1.2)):

$$M_1\sin\varphi + M_2\cos\varphi = M_1^*, \quad M_1 p + M_2 q = M_2^*, \quad M_3 = M_3^*$$
$$M_i^* = M_i^*(G_z, H, r, \tau, \theta), \quad i = 1,2,3 \tag{2.2}$$

To be specific, the case

$$M_1 = pf, \quad M_2 = qf, \quad M_3 = M_3^*, \quad f = f(G_z, H, r, \theta, \tau) \tag{2.3}$$

is considered.

The necessary and sufficient conditions for identities (2.2) to hold are henceforth assumed to be satisfied or, in particular, the sufficient conditions (2.3) that ensure that relations (2.2) hold. System (2.1) can then be represented in the form

$$\dot{G}_z = \varepsilon F_1, \quad \dot{H} = \varepsilon F_2, \quad \dot{r} = \varepsilon F_3$$
$$F_1 = M_1^*\sin\theta + M_3^*\cos\theta, \quad F_2 = M_2^* + M_3^* r, \quad F_3 = C^{-1}M_3^* \tag{2.4}$$

Here, $F_i = F_i(G_Z, H, r, \tau, \theta)$ $(i = 1, 2, 3)$ are 2π-periodic functions of the phase of the angle θ.

It is assumed that the investigation of the perturbed motion is carried out for the slow variables u_i $(i = 1, 2, 3)$. The slow variables G_Z, H and r can be expressed in terms of u_i (1.6) in the following way:[5,14]

$$G_z = \chi\delta\left(u_1 + u_2 + u_3 + u_1u_2u_3 + R\right)^{1/2}$$
$$H = \frac{1}{2}\mu[(u_1 + u_2 + u_3)(1 + AC^{-1}) + (R - u_1u_2u_3)(1 - AC^{-1})]$$
$$r = C^{-1}\chi\left(u_1 + u_2 + u_3 + u_1u_2u_3 - R\right)^{1/2}$$
$$R = \operatorname{sign}(G_z^2 - C^2r^2)[(1 - u_1^2)(1 - u_2^2)(u_3^2 - 1)]^{1/2}$$
$$\chi = \operatorname{sign} r\,(A\mu)^{1/2}, \quad \delta = \operatorname{sign}\left(1 + u_1u_2 + u_1u_3 + u_2u_3\right) \tag{2.5}$$

The signs of χ and R at the initial instant are determined using the initial conditions for G_Z and r. If, in the course of the motion, one or both of the quantities $G_Z^2 - C^2r^2$ and r pass through zero, a change in the signs of χ and R is possible for the determination of which the original system (2.4) can be used.

After a number of transformations, the sought system of equations for the slow variables u_i $(i = 1, 2, 3)$ takes the form

$$\frac{du_i}{dt} = \varepsilon V_i(u_1, u_2, u_3, \tau, \theta), \quad u_i(0) = u_i^0$$
$$V_i = V_{i1}F_1^* + V_{i2}F_2^* + V_{i3}F_3^*, \quad V_{ij} = V_{ij}\left(u_1, u_2, u_3\right), \quad i, j = 1,2,3 \tag{2.6}$$

$$V_{11} = \frac{G_z - Cru_1}{A\,\Delta}, \quad V_{12} = \frac{u_1^2 - 1}{\Delta}, \quad V_{13} = \frac{C}{\Delta}[(CA^{-1} - 1)ru_1^2 - G_zA^{-1}u_1 + r]$$
$$\Delta = \mu\left(u_1 - u_2\right)\left(u_1 - u_3\right) \tag{2.7}$$

The functions V_{2j} and V_{3j} $(j = 1, 2, 3)$ are obtained from the corresponding expressions (2.7) for the same value of j by the cyclic permutation of the subscripts on the quantity u_i. The functions F_i^* are obtained by substituting expressions (2.5) into the functions F_i (2.4). The initial values of the variables u_i are calculated from the initial data G_z^0, H^0, r^0 using relations (1.5).

We substitute the fast variable θ from expression (1.3) for the unperturbed motion into the right-hand sides of system (2.6). The right-hand sides of system (2.6) will then be periodic functions of t with a period $2K(k)/\alpha$, where k and α are defined by relations (1.3). Averaging the right-hand sides of the resulting system over the phase of the nutation angle, we obtain the averaged system of the first approximation $(\tau = \varepsilon t)$

$$\frac{du_i}{d\tau} = U_i\left(u_1, u_2, u_3, \tau\right), \quad u_i(0) = u_i^0; \quad i = 1,2,3$$
$$U_i\left(u_1, u_2, u_3, \tau\right) = \frac{\alpha}{2K(k)}\int_0^{2K/\alpha} V_i\left(u_1, u_2, u_3, \tau, \theta(t)\right)dt \tag{2.8}$$

After studying and solving system (2.8) for u_i, the initial variables G_Z, H and r are recovered using formulae (2.5). The slow variables u_i and G_Z, H and r are determined with an error of the order of ε.

3. Motion of a rigid body under the action of a linear dissipative torque

We now consider a perturbed motion, close to the Lagrange case, under the action of an external medium. An external medium that slowly changes the viscous properties due to a variation in density, temperature and composition of the medium can serve as an example. The perturbed torques εM_i (i = 1, 2, 3) have the form[18]

$$M_1 = -a(\tau)p, \quad M_2 = -a(\tau)q, \quad M_3 = -b(\tau)r; \quad a(\tau), \quad b(\tau) > 0, \quad \tau = \varepsilon t \tag{3.1}$$

where $a(\tau)$ and $b(\tau)$ are integrable functions that depend on the medium properties and the body shape.

Torques (3.1) satisfy conditions (2.2)–(2.4) which makes it possible to average over the phase of the nutation angle θ. For given perturbations, system (2.4) is written in the following way:

$$\begin{aligned} \dot{G}_z &= -\varepsilon\left[(a(\tau)p\sin\varphi + a(\tau)q\cos\varphi)\sin\theta + b(\tau)r\cos\theta\right] \\ \dot{H} &= -\varepsilon[a(\tau)(p^2+q^2) + b(\tau)r^2] \\ \dot{r} &= -\varepsilon C^{-1}b(\tau)r \end{aligned} \tag{3.2}$$

The third equation of (3.2) can be integrated:

$$r = r^0 \exp\left(-\varepsilon C^{-1}\int_0^t b(\varepsilon t)dt\right) \tag{3.3}$$

We now consider the case when

$$\begin{aligned} &a(\tau) = a_0 + a_1\tau, \quad b(\tau) = b_0 + b_1\tau \\ &a_0, a_1, b_0, b_1 - \text{const}, \quad a_0 > 0, \quad b_0 > 0, \quad a_1 \ge 0, \quad b_1 \ge 0 \end{aligned} \tag{3.4}$$

After a number of transformations, averaged system (2.8), taking account of torques (3.1), takes the form

$$\begin{aligned} \frac{du_1}{d\tau} &= \frac{-1}{A\Delta}\{a(\tau)[A^{-1}(G_z - C\,r\,u_1)(G_z - Cr\,\upsilon) \\ &+ (u_1^2 - 1)(2H - Cr^2 - 2\mu\upsilon)] + b(\tau)\,r\,(G_z - C\,r\,u_1)(\upsilon - u_1)\}\ (123) \\ \upsilon &= u_3 - (u_3 - u_1)\,E(k)/K(k) \end{aligned} \tag{3.5}$$

The symbol (123) denotes that the two unwritten relations are obtained from the written relation by cyclic permutation of the indices 1, 2, 3. $K(k)$ and $E(k)$ are complete elliptic integrals of the first and second kind and, instead of G_z, H, r and k, the expressions for them (2.5) and according to the last formula of (1.3) are substituted.

Averaged system (3.5) was numerically integrated for various initial conditions and values of the problem parameters. Three cases, corresponding to the initial data presented in Table 1, are considered.

It is assumed that, at the initial instant t = 0, a Lagrange top has acquired an angular velocity of rotation about the axis of dynamic symmetry equal to $r^0 = \sqrt{3}$ and, moreover,

$$A = 1.5, \quad C = 1, \quad \mu = 0.5, \quad a_1 = b_1 = 1, \quad u_2^0 = \cos\theta^0$$

At the initial instant, the angle of deviation of the dynamic symmetry axis from the vertical is equal to θ^0. Graphs of the functions G_z, H, r and u_i (i = 1, 2, 3), obtained as the result of numerical calculation, are shown in Fig. 1 for the above mentioned three cases.

In Case 1, the total energy of the body, the projection of the angular momentum vector onto the vertical and the angular velocity of rotation about the dynamic symmetry axis decrease. The quantity u_3 tends fairly rapidly to unity and the variables u_1 and u_2 tend to −1. It follows from the first equality of (1.3) that, in this case, $\cos\theta \to -1$ when $\theta \to \pi$.

In Case 2, the variable u_3 also tends to unity but the interval of monotonic decrease increased. The projection of the angular momentum vector G_z and the value of r monotonically decrease to zero. The total energy H monotonically decreases, approaching the value H = −0.5.

In Case 3, the projection of the angular momentum vector onto the vertical G_z tends to zero but, unlike in Cases 1 and 2, monotonically increases. The total energy of the body tends fairly rapidly to −0.5. The values of r and u_3 monotonically decrease, and the functions u_1 and u_2 also monotonically decrease, as is seen from the large scale plots presented in the upper right corner of the bottom part of Fig. 1.

Table 1

Case	u_1^0	u_2^0	u_3^0	θ^0
1	0.913	0.996	1.087	5°
2	0	0.5	2	60°
3	−0.932	−0.866	2.932	150°

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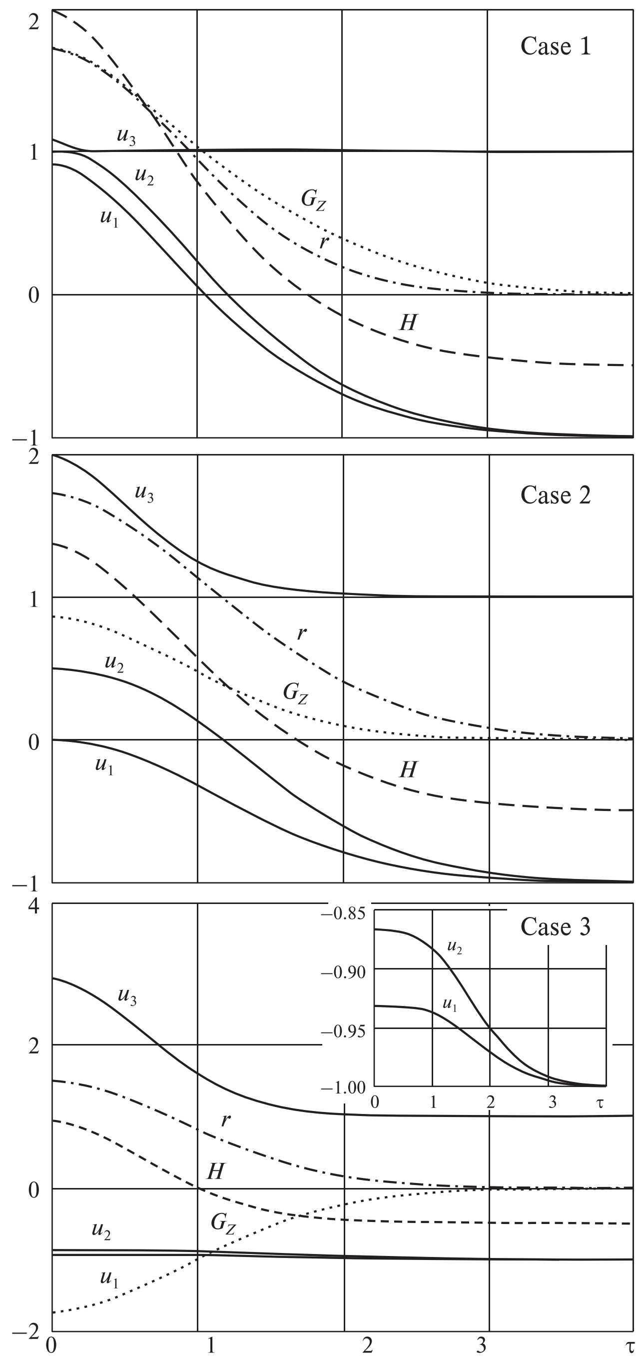

Fig. 1.

4. Conclusion

In comparing the results obtained with the results in Refs 5 and 14, where M_i are independent of the slow time τ, their distinct mechanical content can be stressed: the dependence of the perturbing torque on the slow time leads to the appearance, in the averaged system of equations of the first approximation for the slow variables, of the functions $a(\tau)$ and $b(\tau)$ depending on the slow time which, in the case of numerical integration, smooth out the behaviour of u_i ($i = 1, 2, 3$), G_z and H. Under the action of dissipative torque (3.1), the body tends to a stable lower equilibrium position more rapidly than in the case considered earlier[5,14] which follows from the specification of coefficients (3.4).

The correctness of the calculation is confirmed by the fact that the values of r obtained using the numerical data and formulae (2.5) are practically identical to exact solution (3.3).

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