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CONTROL IN DETERMINISTIC SYSTEMS

Quasi-Optimal Deceleration of Rotations of a Rigid Body with a Moving Mass in a Resistive Medium

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Abstract—The problem of time quasi-optimal deceleration of the rotations of a dynamically symmetric rigid body is studied. It is assumed that the body contains a point mass connected to it with a strong viscoelastic element (damper). The body is acted on by a small linear resistance torque of the medium that is proportional to the angular momentum and a small control torque bounded by an ellipsoidal domain. An approximate synthesis of control is proposed, and an asymptotic solution based on a procedure of averaging the precession motion over the phase is obtained; numerical integration is performed. The main properties of the quasi-optimal motion are determined.

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INTRODUCTION

The analysis of hybrid systems, i.e., systems containing elements with distributed and lumped parameters is of interest both for theory and applications. For systems containing quasi-rigid bodies, various approaches were developed and constructive results were obtained. The models of quasi-rigid bodies assume that their motion is in some sense close to the motion of rigid bodies. The compliance is taken into account by introducing additional terms into Euler's equations of motion for a fictitious rigid body. Rigid bodies with internal degrees of freedom were studied in a number of publications (e.g., see [1–5]).

Considerable attention was paid to the analysis of the passive motion of a rigid body in a resistive medium [2, 4, 6]. The problem of controlling the rotation of quasi-rigid bodies using concentrated moments of forces, which is important for applications, has been studied less [3, 4, 7].

In this paper, we consider the quasi-optimal (close to time optimal) control of the deceleration of rotations of a dynamically symmetric rigid body with a moving mass attached by a viscoelastic damper to a point on the symmetry axis. The rigid body is acted on by a small moment of forces of the medium's resistance. The components of the control torques are represented by the products $\varepsilon b_i u_i$ ($i = 1, 2, 3$), where εb_i characterizes the efficiency of the control system with respect to the ith axis, b_i are constants that are close to each other (optimality is achieved when they are identical), and u_i are dimensionless control functions in the form of feedbacks. The magnitudes of the control and dissipative torques $O(\varepsilon)$ are small compared to the initial kinetic energy of the body (the dimensionalities of these quantities are the same).

1. STATEMENT OF THE PROBLEM

We consider the controlled rotation of a dynamically symmetric rigid body with a moving point mass attached by a strong viscoelastic damper to a point on the symmetry axis (in the undeformed state) [1, 2] in a resistant medium. Under the approach described in [3], the asymptotically approximate equations of the controlled rotation in the reference frame fixed to the body (Euler's equations) have the form

$$\dot{\mathbf{G}} + \boldsymbol{\omega} \times \mathbf{G} = \mathbf{M}^u + \mathbf{M}^\nu + \mathbf{M}^r. \tag{1.1}$$

Here, $\mathbf{M}^u$ is the vector of the control external (reactive) moment of forces, $\mathbf{M}^v$ is the vector of the internal disturbing moment of forces caused by the presence of the viscoelastic element, and $\mathbf{M}^r$ is the moment of the medium's resistance forces [4]. The vector $\mathbf{G} = \mathbf{J}\boldsymbol{\omega}$ is the angular momentum of the body, where $\mathbf{J} = \operatorname{diag}(A_1, A_1, A_3)$ is the tensor of inertia of the undisturbed body reduced to the principal axes, and $\boldsymbol{\omega} = (p, q, r)$ is the vector of the angular velocity represented by its projections on the axes coinciding with the principal central axes of inertia. The magnitude of the body's angular momentum is

$$G = |\mathbf{G}| = [A_1^2(p^2 + q^2) + A_3^2 r^2]^{1/2}, \; A_1 \neq A_3.$$

Let us describe the control structure. The magnitude of the control torque is assumed to be of order ε, where $\varepsilon \ll 1$ is a small parameter. Its components are represented in the form (see [3, 5])

$$M_i^u = \varepsilon b_i u_i, \tag{1.2}$$

$$u_i = -G_i G^{-1}, \quad i = 1, 2, 3, \; |\mathbf{u}| = 1.$$

To simplify the solution of the optimal control problem, we impose a structural constraint on system (1.1)—we assume that the moment of the resistance forces is small and proportional to the angular momentum (see [6]):

$$\mathbf{M}^r = -\varepsilon\lambda\mathbf{J}\boldsymbol{\omega}; \tag{1.3}$$

here λ is a constant proportionality factor, which is mainly determined by the properties of the medium and the shape of the body; this factor has the dimensionality of the angular velocity.

Taking into account Eqs. (1.2) and (1.3), the approximate system of equations (1.1) of the controlled motion in the projections on the principal central axes of inertia has the form (see [1–6])

$$\begin{gathered}
A_1\dot{p} + (A_3 - A_1)qr = -\varepsilon b_1 \frac{A_1 p}{G} + FG^2 qr + Dr^4 p - \varepsilon\lambda A_1 p, \\
A_1\dot{q} + (A_1 - A_3)pr = -\varepsilon b_2 \frac{A_1 q}{G} - FG^2 pr + Dr^4 q - \varepsilon\lambda A_1 q, \\
A_3\dot{r} = -\varepsilon b_3 \frac{A_3 r}{G} - A_1 A_3^{-1} D r^3 (p^2 + q^2) - \varepsilon\lambda A_3 r, \\
0 < A_3 \leq 2A_1, \; A_3 \neq A_1.
\end{gathered} \tag{1.4}$$

Note that if the coefficients $b_1 = b_2 = b_3$ are all equal, control (1.2) is optimal for all values of ε; this property explains the simplifying assumption about the closeness of b_i and the introduction of the term *quasi-optimal control*. The quantities F and D introduced in (1.4) are expressed in terms of the system parameters as follows (see [1, 2]):

$$F = m\rho^2\Omega^{-2}A_3 A_1^{-3}, \; D = m\rho^2\Lambda\Omega^{-4}A_3^3(A_1 - A_3)A_1^{-4}. \tag{1.5}$$

The coefficients D and F in (1.5) characterize the disturbing moments of forces due to the presence of the viscoelastic element, m is the mass of the moving point, and ρ is the distance from the center of mass of the undeformed system to the attachment point, which is on the dynamic symmetry axis of the body. The constants $\Omega^2 = c/m$ and $\Lambda = \delta/m$ determine the frequency of the oscillations and the rate of their damping, respectively; c is the rigidity (the coefficient of elasticity); and δ is the viscosity coefficient of the damper. We consider the case of a strong damper when the inequalities (see [1, 2])

$$\Omega^2 \gg \Lambda\omega_0 \gg \omega_0^2, \tag{1.6}$$

where ω_0 is the magnitude of the initial value of the angular velocity vector, are satisfied.

Inequalities (1.6) allow us to introduce a small parameter into (1.5) and assume that the disturbing torques are small so that the averaging procedure can be applied after a certain initial transient process.

We state the problem of time quasi-optimal deceleration of the rotations

$$\boldsymbol{\omega}(T) = 0, \, T \to \min_{\mathbf{u}}, \, |\mathbf{u}| \leq 1, \tag{1.7}$$

which assumes that the parameters b_i $(b_i \approx b, |b_i - b| \ll b)$ are close to each other.

2. SOLUTION OF THE QUASI-OPTIMAL DECELERATION PROBLEM

To solve the quasi-optimal deceleration problem, it is convenient to reduce it to a dimensionless form. As the characteristic parameters of the problem, we use, for definiteness, the moment of inertia of the rigid body about the axis $x_1 - A_1 = A_2$, and the order of the initial velocity (see Section 3) ω_0. Define the dimensionless coefficients of inertia $\tilde{A}_i = A_i/A_1$ and the dimensionless time $\tau = \omega_0 t$. Then, system (1.4) takes the form

$$\frac{d\tilde{p}}{d\tau} = -\left(\tilde{A}_3 - 1\right)\tilde{q}\tilde{r} - \varepsilon\tilde{b}_1\tilde{p}/\tilde{G} + \varepsilon\tilde{F}\tilde{G}^2\tilde{q}\tilde{r} + \varepsilon\tilde{D}\tilde{r}^4\tilde{p} - \varepsilon\tilde{\lambda}\tilde{p},$$

$$\frac{d\tilde{q}}{d\tau} = -\left(1 - \tilde{A}_3\right)\tilde{p}\tilde{r} - \varepsilon\tilde{b}_2\tilde{q}/\tilde{G} - \varepsilon\tilde{F}\tilde{G}^2\tilde{p}\tilde{r} + \varepsilon\tilde{D}\tilde{r}^4\tilde{q} - \varepsilon\tilde{\lambda}\tilde{q}, \tag{2.1}$$

$$\frac{d\tilde{r}}{d\tau} = -\varepsilon\tilde{b}_3\tilde{r}/\tilde{G} - \varepsilon\tilde{A}_3^{-1}\tilde{D}\tilde{r}^3\left(\tilde{p}^2 + \tilde{q}^2\right) - \varepsilon\tilde{\lambda}\tilde{r}.$$

Here, based on the assumptions made above, we introduced the notation

$$\varepsilon\tilde{F} = m\rho^2\Omega^{-2}\tilde{A}_3 A_1^{-1}\omega_0^2,\ \varepsilon\tilde{D} = m\rho^2\Lambda\Omega^{-4}\tilde{A}_3^3\left(1 - \tilde{A}_3\right)A_1^{-1}\omega_0^3,$$

$$\varepsilon\tilde{b}_i = b_i/(A_1\omega_0^2),\ \varepsilon\tilde{\lambda} = \lambda/\omega_0,\ \tilde{G} = G/(A_1\omega_0),\ \tilde{A}_1 = \tilde{A}_2 = 1.$$

Let us use the general generating solution to system (2.1) for $\varepsilon = 0$:

$$\tilde{r} = \text{const},\ \tilde{p} = \tilde{a}\cos\psi,\ \tilde{q} = \tilde{a}\sin\psi,\ \tilde{a} > 0, \quad \text{const} \neq 0. \tag{2.2}$$

Here, $\psi = (\tilde{A}_3 - 1)\tilde{r}\tau + \psi_0$ is the phase of the oscillation of the equatorial component of the angular velocity vector.

Substitute (2.2) into the third equation in (2.1) and average the resulting system of equations for $\tilde{a}$, $\tilde{r}$. Introducing the slow argument $\theta = \varepsilon\tau$, we obtain $(' = d/d\theta)$

$$\tilde{a}' = -\frac{\tilde{a}}{2}\left[\tilde{G}^{-1}\left(\tilde{b}_1 + \tilde{b}_2\right) - 2\tilde{D}\tilde{r}^4 + 2\tilde{\lambda}\right], \tag{2.3}$$

$$\tilde{r}' = -\tilde{r}\left(\tilde{b}_3\tilde{G}^{-1} + \tilde{A}_3^{-2}\tilde{D}\tilde{r}^2\tilde{a}^2 + \tilde{\lambda}\right).$$

The mean value of the expressions containing the factor $\tilde{F}$ is zero.

Note that, in the case $\tilde{b}_1 = \tilde{b}_2 = \tilde{b}_3$, the equations for $\tilde{a}$ and $\tilde{r}$ are completely integrable, and this problem was solved analytically in [4, 7].

Consider the particular case

$$1/2(\tilde{b}_1 + \tilde{b}_2) = \tilde{b}_3 = \tilde{b}. \tag{2.4}$$

Multiply the first equation in (2.1) by $\tilde{p}$, the second one by $\tilde{q}$, the third equation by $\tilde{A}_3^2\tilde{r}$, and add them. Upon averaging, we obtain the equation

$$\tilde{G}' = -\tilde{b} - \tilde{\lambda}\tilde{G}.$$

This equation should be integrated. Taking into account the initial condition $\tilde{G}(\theta_0) = \tilde{G}^0$ and the terminal condition $\tilde{G}\left(T, \theta_0, \tilde{G}^0\right) = 0$, $T = T\left(\theta_0, \tilde{G}^0\right)$, we obtain

$$\tilde{G}(\theta) = -\frac{\tilde{b}}{\tilde{\lambda}} + \left(\tilde{G}^0 + \frac{\tilde{b}}{\tilde{\lambda}}\right)\exp\left(-\tilde{\lambda}\theta\right),\ \Theta = \frac{1}{\tilde{\lambda}}\ln\left(\tilde{G}^0\frac{\tilde{\lambda}}{\tilde{b}} + 1\right). \tag{2.5}$$

Note that $\Theta \to \infty$ as $\tilde{G}^0/\tilde{b} \to \infty$ for all $\tilde{\lambda}$; conversely, $\Theta \to 0$ as $\tilde{G}^0\tilde{\lambda}/\tilde{b} \to 0$ ($\tilde{\lambda}$ is arbitrary) or as $\tilde{\lambda} \to \infty$.

For system (2.3) under condition (2.4), we can make the change of variables $\tilde{r} = \eta\tilde{G}$, $\tilde{a} = \alpha\tilde{G}$. Then, Eqs. (2.3) take the form

$$\alpha' = \alpha\eta^4\tilde{G}^4\tilde{D},\ \eta' = -\alpha^2\eta^3\tilde{A}_3^{-2}\tilde{D}\tilde{G}^4. \tag{2.6}$$

Divide the first equation by the second one to obtain

$$\frac{d\alpha}{d\eta} = -A_3^2\eta/\alpha.$$

We have the first integral C_1:

$$\eta^2 = 2C_1 - \tilde{A}_3^{-2}\alpha^2,\ C_1 = \frac{1}{2}\tilde{A}_3^{-2}. \tag{2.7}$$

Substitute η^2 from (2.7) into the first equation in (2.6) to obtain

$$\alpha' = \alpha\left(\tilde{A}_3^{-2} - \tilde{A}_3^{-2}\alpha^2\right)^2\tilde{D}\tilde{G}^4,\ \tilde{G}_0 = 1. \tag{2.8}$$

Upon the substitution of expression (2.5) for $\tilde{G}$ into Eq. (2.8) for α, this equation can be integrated, and its solution is (see [8])

$$\begin{aligned}\frac{1}{2\tilde{A}_3^{-2}\left(\tilde{A}_3^{-2} - \tilde{A}_3^{-2}\alpha^2\right)} + \frac{1}{2\tilde{A}_3^{-4}}\ln\left|\frac{\alpha^2}{\tilde{A}_3^{-2} - \tilde{A}_3^{-2}\alpha^2}\right| = \tilde{D}\Bigg[\frac{\tilde{b}^4}{\tilde{\lambda}^4}\theta \\ + \frac{4\tilde{b}^3}{\tilde{\lambda}^4}b_*\exp\left(-\tilde{\lambda}\theta\right) - \frac{3\tilde{b}^2}{\tilde{\lambda}^3}\alpha b_*^2\exp\left(-2\tilde{\lambda}\theta\right) \\ + \frac{4\tilde{b}}{3\tilde{\lambda}^2}b_*^3\exp\left(-3\tilde{\lambda}\theta\right) - \frac{1}{4\tilde{\lambda}}b_*^4\exp\left(-4\tilde{\lambda}\theta\right)\Bigg] + C_2, \quad b_* = 1 + \frac{\tilde{b}}{\tilde{\lambda}}.\end{aligned} \tag{2.9}$$

The second constant of integration C_2 is

$$\begin{aligned}C_2 = \frac{1}{2\tilde{A}_3^{-2}\left(\tilde{A}_3^{-2} - \tilde{A}_3^{-2}\alpha_0^2\right)} + \frac{1}{2\tilde{A}_3^{-4}}\ln\left|\frac{\alpha_0^2}{\tilde{A}_3^{-2} - \tilde{A}_3^{-2}\alpha_0^2}\right| \\ - \tilde{D}\left[\frac{4\tilde{b}^3}{\tilde{\lambda}^4}b_* - \frac{3\tilde{b}^2}{\tilde{\lambda}^3}b_*^2 + \frac{4\tilde{b}}{3\tilde{\lambda}^2}b_*^3 - \frac{1}{4\tilde{\lambda}}b_*^4\right].\end{aligned}$$

The quantities ρ and α are related by Eq. (2.7). Thus, we have obtained expressions for the parameters of the optimal motion $\tilde{G}(\theta)$, Θ (2.5) and $a(\theta)$, $r(\theta)$ (2.6)–(2.9). Their qualitative properties are simple and easy to obtain.

3. NUMERICAL CALCULATIONS

System (2.3) was solved numerically for the renormalized initial conditions $\tilde{G}_0 = 1$, $\tilde{A}_3 = 1.2$, $\tilde{a}_0 = 0.35$, and $\tilde{r}_0 = (1 - \tilde{a}_0^2)^{1/2}/\tilde{A}_3^2$; the values of the resistance coefficient $\tilde{\lambda} = 1.2; 1.8$; the coefficients of the control torque $\tilde{b}_1 = 1.625$, $\tilde{b}_2 = 1$, $\tilde{b}_3 = 1.25$ with $(\tilde{b}_1 + \tilde{b}_2)/2 \neq \tilde{b}_3$; and the coefficient $\tilde{D} = 1$. In the first calculation (curve *1*), $\tilde{\lambda} = 1.2$. In the second calculation (curve *2*), $\tilde{\lambda} = 1.8$. The parameters were chosen

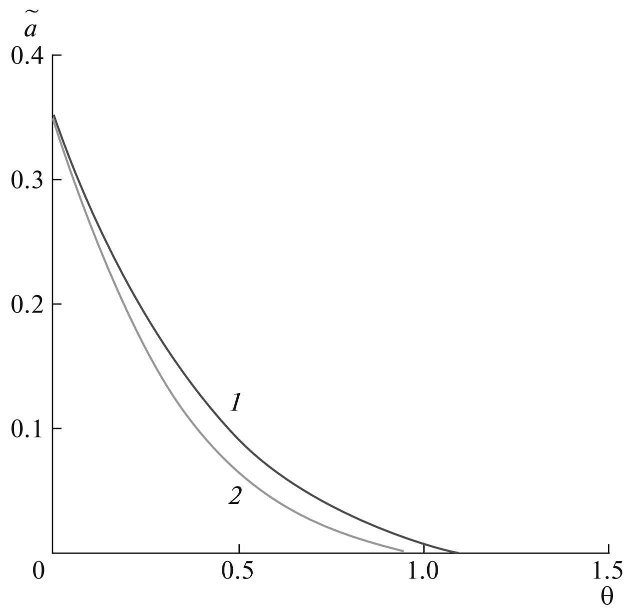

Fig. 1.

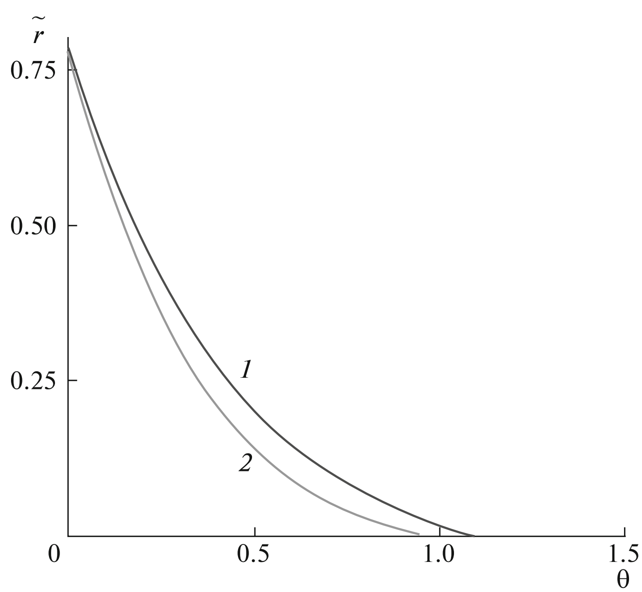

Fig. 2.

such that they satisfy the conditions $\tilde{A}_3 \leq 2$ and $\tilde{a}_0 < \tilde{r}_0$. To construct the plot of the magnitude of the angular momentum, the expression

$$\tilde{G} = \left|\tilde{\mathbf{G}}\right| = \left[\tilde{a}^2 + \tilde{A}_3^2\tilde{r}^2\right]^{1/2}$$

was used. Figures 1–3 show the plots of the functions $\tilde{a}$, $\tilde{r}$, and $\tilde{G}$, which are similar in shape.

It is seen from these plots (Fig. 3, curves *1*, *2*) that the deceleration rate of the body is naturally higher when the resistance coefficient increases (curve *2*). The deceleration time is $T = 1.1$ in the first case and $T = 0.96$ in the second case (Figs. 1–3). Note that the evolution of the magnitude of the angular momentum does not change when the coefficients b_i are close to each other (the quadratic effect). The influence of the internal forces due to the displacement of the moving mass is small; for this reason, the variation of the coefficient $\tilde{D}$ is not shown.

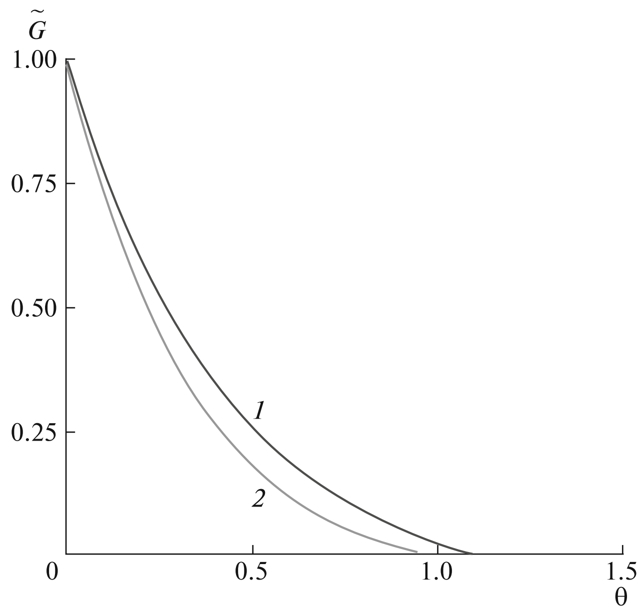

Fig. 3.

CONCLUSIONS

The time quasi-optimal deceleration of the rotation of a dynamically symmetric rigid body with a viscoelastic element in a resistant medium is investigated. Within the asymptotic approach, an averaged system of equations is derived; for the chosen numerical values of the dimensionless parameters, the deceleration time $T = 0.96$ and $T = 1.1$ is found; and the plots of the changing angular momentum and the magnitudes $\tilde{a}$ and $\tilde{r}$ of the equatorial and axial components of the angular velocity vector of the quasi-rigid body are constructed; these plots are similar in shape.

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