THE PROPAGATION OF HARMONIC HEAT WAVES IN A PERODICAL SYSTEM OF PUNCHES ON A HALF PLANE

V.Gavdzinski, M.G.El-Sheikh, E.Maltseva
Odessa State Academy of Structures and Architecture,
Kuwait University, Economical University

The method of discrete Riemann's problem was originally proposed [1] for solving finite mixed steady problems. By gradual nodification and amplifications, it has become of wide applications to solutions of problems in several branches of mathematical physics [2].

In this work, the propagation of harmonic heat waves in a periodical system of punches on a half plane is considered. The thermoelastic punch problems occurring in engineering mathematics, and a priori the corresponding problem of heat conductivity, have become of major interest in recent investigations.

The point of departure is the following heat equation

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=\frac{1}{a_T}\cdot\frac{\partial T}{\partial t} \quad (y>0) \tag{1}$$

with mixed boundary conditions

$$T(x,0,t)=e^{-i\omega t}f(x)\,,\ \text{if}\quad x\in\Delta_1 \tag{2}$$

$$\frac{\partial T(x,0,t)}{\partial y}=0\,,\quad \text{if}\quad x\in\Delta_2 \tag{3}$$

$$\lim_{y\to+\infty}|T(x,y,t)<+\infty\,, \tag{4}$$

where $\Delta_1=[-a,a]$, $\Delta_2=[-\pi,\pi]\setminus\Delta_1$, a_T is the thermal diffusivity. The mixed conditions (2) and (3) are periodically continued over the whole x-axis with a period 2π as a reflection to the periodicity of the punches throughout the half-plane.

The mixed boundary conditions (2), (3) can be replaced by the two uniform and compatible ones:

$$T^*(x,0)=f_-(x)+\varphi_+(x),\quad \frac{\partial T^*(x,0)}{\partial y}=\varphi_-(x) \tag{5}$$

where

$$\begin{aligned}
\varphi_+(x) &= \begin{cases} 0, & x\in\Delta_1,\\ \textit{undetermined}, & x\in\Delta_2,\end{cases}\\
\varphi_-(x) &= \begin{cases} \textit{undetermined}, & x\in\Delta_1,\\ 0, & x\in\Delta_2,\end{cases}\\
f_-(x) &= \begin{cases} f(x), & x\in\Delta_1,\\ 0, & x\in\Delta_2\end{cases}
\end{aligned} \tag{6}$$

$$T(x,y,t)=e^{-i\omega t}T^*(x,y)$$

Applying finite Fourier transform with respect to x to the boundary conditions (5) and the equation (1) we arrive at the following discrete Riemann's problem

$$\Phi_{n+}=-\lambda_n^{-1}\Phi_{n-}-F_{n-}\quad(n=0,\pm1,\pm2,...) \tag{7}$$

where

$$\Phi_{n\pm}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi_{\pm}e^{-inx}dx,\; F_{n-}=\frac{1}{2\pi}\int_{-\pi}^{\pi}f_{-}e^{-inx}dx,\; \lambda^2=n^2-\frac{i\omega}{a_T}$$

Multiplying (7) by n, we rewrite it in such a form

$$n\Phi_{n+}=-\operatorname{sgn}\left(n+\frac{1}{2}\right)\Phi_{n-}+\Gamma_n\Phi_{n-}-n\Phi_{n-},\;(n=\pm1,\pm2,...) \tag{8}$$

where $\Gamma_n=\operatorname{sgn}\left(n+\frac{1}{2}\right)-n\lambda_n^{-1}$ and also $|\Gamma_n|=O\left(\frac{1}{n^2}\right)\;(n\to\infty)$

Additionally, the condition

$$-\sum_{n=-\infty}^{+\infty}\lambda_n^{-1}\Phi_{n-}=f(0) \tag{9}$$

determines the solution of problem (8) equivalent to that of (7). Performing the inverse Fourier transform

$$W^{-1}\Phi_{n+}=\sum_{n=-\infty}^{+\infty}\Phi_{n\pm}e^{inx}=\varphi_{\pm}(x) \tag{10}$$

we reduce the discrete problem (8) to the singular integral equation

$$\frac{1}{\pi}\int_{-a}^{a}\frac{\varphi_{-}(\xi)d\xi}{1-e^{i(x-\xi)}}=\frac{1}{2\pi}\int_{-a}^{a}\gamma(x-\xi)\varphi_{-}(\xi)d\xi+if'(x) \tag{11}$$

The equation (11) we solve approximately. To do this we use Cherskii theorem [3].

THEOREM. *Let the following conditions be fulfilled:*

1. *The approximate equation $\widetilde{K}\widetilde{\varphi}=\widetilde{f}$ has the unique solution.*
2. $f-f\in Y_0$, *where Y_0 is a linear subset, $Y_0\subset Y$.*

3. *Operator* $K-\widetilde{K}$ *is acting from* X *into* Y_0.

4. *On* Y_0 *the inverse operator* $\widetilde{K}^{-1}$ *is determined acting from* Y_0 *into* X_0.

5. $\left\|\widetilde{K}^{-1}(K-\widetilde{K})\right\|<1$

Then the equation $K\varphi=f$ has a unique solution equals to

$$\varphi=\widetilde{\varphi}+\left[I+\widetilde{K}^{-1}(K-\widetilde{K}\right]^{-1}\widetilde{K}^{-1}(f-K\varphi)$$

and the following estimate holds:

$$\left\|\varphi-\widetilde{\varphi}\right\|_{X_0}\leq\frac{\left\|\widetilde{K}^{-1}(f-K\widetilde{\varphi})\right\|_{X_0}}{1-\left\|\widetilde{K}^{-1}(K-\widetilde{K})\right\|}$$

Inverting the Cauchy type integral on the left of the equation (11) we have [4]

$$K\varphi_-\equiv\varphi_-(x)-\frac{1}{2\pi^2R(x)}\int_{-a}^{a}\frac{R(\xi)e^{i\xi}d\xi}{e^{i\xi}-e^{ix}}\int_{-a}^{a}\gamma(\xi-y)\varphi_-(y)dy=g(x)\quad(12)$$

where

$$g(x)=\frac{1}{R(x)}[m(x)+a_0],\quad m(x)=-\frac{1}{\pi i}\int_{-a}^{a}\frac{f'(\xi)R(\xi)e^{i\xi}}{e^{i\xi}-e^{ix}}d\xi$$

$$R(x)=-e^{ix/2}\sqrt{2(\cos x-\cos a)}$$

Let now $f'\in L_r[-a,a]$; $r>4/3$, then $g(x)\in L_p[-a,a]$; $1<p<4/3$

Therefore we put $X=X_0=Y=Y_0=L_p[-a,a]$

The norm of the element $\varphi_-(x)$ in the space L_p is defined by the way

$$\left\|\varphi_-\right\|_{L_p}=\left(\int_{-a}^{a}|\varphi_-(x)|^p\,dx\right)^{1/p}\quad(13)$$

The operator $\widetilde{K}$ we determine by the equality

$$K\widetilde{\varphi}_-\equiv\widetilde{\varphi}_-(x)-\frac{1}{2\pi^2R(x)}\int_{-a}^{a}\frac{R(\xi)e^{i\xi}}{e^{i\xi}-e^{ix}}d\xi\int_{-a}^{a}\sum_{n=-N}^{N}\Gamma_ne^{in(\xi-y)}\widetilde{\varphi}_-(y)dy\quad(14)$$

LEMMA. *For any given* $\varepsilon > 0$ *we have* $\left\| K - \widetilde{K} \right\| < \varepsilon$ *under appropriate choice of* N. *Estimating the norm* $\left\| K - \widetilde{K} \right\|$ *we obtain*

$$\left\| K - \widetilde{K} \right\| \leq \frac{(1 + |\cos a|)\pi^{1/p}\Gamma^{1/p}(2-p)}{2^{2/p}\Gamma^{2/p}\left(\frac{3}{2} - \frac{p}{2}\right)} \sum_{|k|>N} |\Gamma_k| = \Theta(N) \quad (15)$$

By virtue of $|\Gamma_k| = O\left(\frac{1}{k^2}\right)$ as $k \to \infty$ the series $\sum_{k=\infty}^{+\infty} |\Gamma_k|$ converges and so that $\left\| K - \widetilde{K} \right\| < \varepsilon$ under appropriate choice of N. The lemma has been proved.

To find the exact solution of the approximate equation

$$\widetilde{K}\widetilde{\varphi}_{-} = \widetilde{g}(x) \quad (16)$$

we write it in the form

$$\widetilde{\varphi}_{-}(x) = \frac{1}{R(x)} \sum_{k=-N}^{N} \Gamma_k \widetilde{\Phi}_{k-} \alpha_k(x) + \widetilde{g}(x) \quad (17)$$

where

$$\alpha_k(x) = \begin{cases} -e^{-ikx} \sum_{m=0}^{k+1} \mu_m(\cos a) e^{-i(m-1)x}, & k \geq 0, \\ e^{ikx} \sum_{m=0}^{-k-1} \mu_m(\cos a) e^{imx}, & k < 0, \end{cases} \quad (18)$$

$$\widetilde{\Phi}_{n-} = M_n + \sum_{k=-N}^{N} \Gamma_k \widetilde{\Phi}_{k-} N_{kn} + \widetilde{a}_0 R_n \quad (n = 0, \pm 1, \pm 2, \ldots, \pm N) \quad (19)$$

$$N_{kn} = \begin{cases} -\frac{1}{2}\sum_{m=0}^{k+1} \mu_{k-m+1}(\cos a) P_{m-n-1}(\cos a), & k \geq 0, \\ \frac{1}{2}\sum_{m=0}^{-k-1} \mu_{-k-m-1}(\cos a) P_{m+n+1}(\cos a), & k < 0, \end{cases} \tag{20}$$

$R_n = \frac{1}{2} P_n(\cos a)$, $\mu_n(\cos a) = P_n(\cos a) - 2\cos a P_{n-1}(\cos a) +$

$+ P_{n-2}(\cos a)$, $P_n(\cos a)$ – Legendre polynomials.

Solving the system (19) we get

$$\widetilde{\Phi}_{n-} = \frac{\Delta^{(n)}}{\Delta} = \frac{1}{\Delta}\sum_{j=-N}^{N} \widetilde{G}_j \Delta_j^{(n)} = \frac{1}{2\pi\Delta}\sum_{j=-N}^{N} \Delta_j^{(n)} \int_{-a}^{a} \widetilde{g}(x) e^{-ijx} dx \tag{21}$$

If $\Delta \neq 0$ it is possible to prove that the inverse operator $\widetilde{K}^{-1}$ is bounded and

$$\left\| K - \widetilde{K} \right\| \left\| \widetilde{K}^{-1} \right\| < 1$$

When calculating the coefficients $\widetilde{\Phi}_{n-}$ by the formula (21), the constant $\widetilde{a}_0$ is defined from the expression (9)

$$\widetilde{a}_0 = -\frac{f(0) + \frac{1}{\Delta}\sum_{-N}^{N} \lambda_n^{-1} \sum_{j=-N}^{N} M_j \Delta_j^{(n)}}{\frac{1}{\Delta}\sum_{n=-N}^{N} \lambda_n^{-1} \sum_{j=-N}^{N} R_j \Delta_j^{(n)}} \tag{22}$$

where

$$M_n = \frac{1}{2\pi}\int_{-a}^{a} \frac{m(x) e^{-inx}}{R(x)} dx$$

Therefore we have the following theorem

THEOREM. *Let in the equation* (11) $f'(x) \in L_r[-a,a]$, $r > 4/3$ *and the condition* (9) *is fulfilled. Then this equation according to the Cherskii theorem has the unique solution in the class* L_p.

The function $\widetilde{\varphi}_-(x)$ defined by the formula (17) is the approximate solution of the equation (11) and the following estimate holds

$$\|\varphi-\widetilde{\varphi}_-\|_{L_p} \le \frac{Q(N)\|g-K\widetilde{\varphi}_-\|_{L_p}}{1-\Theta(N)Q(N)} \qquad (23)$$

where

$$Q(N) = 1+\frac{(2a)^{1/q} I_p^{1/p}}{2\pi\Delta}\left[\sum_{n=0}^{N}|\Gamma_n|\sum_{j=-N}^{N}|\Delta_j^{(n)}|\sum_{m=0}^{n+1}|\mu_n(\cos a)| + \sum_{n=-N}^{-1}|\Gamma_n|\sum_{j=-N}^{N}|\Delta_j^{(n)}|\sum_{m=0}^{-n-1}|\mu_m(\cos a)|\right]$$

$$I_p = \int_{-a}^{a}\frac{dx}{|R(x)|^p}$$

Finally the approximate solution of the problem (1)-(4) can be written in the form

$$\widetilde{T}(x,y,t) = e^{-i\omega t}\widetilde{T}^*(x,y) = -e^{-i\omega t}\sum_{-N}^{N}\lambda_n^{-1}\widetilde{\Phi}_{n-}e^{inx-\lambda_n y} \qquad (24)$$

Let $\overline{\omega}=0{,}1$ and $N=20$, where $\overline{\omega}=\frac{\omega}{\omega^*}$, $\omega^*=c_2^2/a_T$.

Below in the tables 1 and 2 the values of $\widetilde{\varphi}_-(\overline{x})$ and its modulus calculated by the formula (17) are given, when $\overline{x}\in[-1,1]$, where $\overline{x}=\frac{c_2}{a_T}x$.

Table 1

$\overline{x}$	0	0,1	0,2	0,3
$\widetilde{\varphi}_-(\overline{x})$	-0,3948+ 0,2177i	-0,3964+ 0,2188i	-0,4013+ 0,2252i	-0,4099+ 0,2282i
$\overline{x}$	0,4	0,5	0,6	0,7

$\widetilde{\varphi}_-(\overline{x})$	-0,4234+ 0,2396i	-0,4435+ 0,2510i	-0,4743+ 0,2713i	-0,5237+ 0,3028i
$\overline{x}$	0,8	0,9	0,95	0,99
$\widetilde{\varphi}_-(\overline{x})$	-0,6148+ 0,3597i	-0,8279+ 0,4928i	-1,1435+ 0,6857i	-2,5073+ 1,5127i

Table 2

$\overline{x}$	0	0,1	0,2	0,3	0,4	0,5
$\widetilde{\varphi}_-(\overline{x})$	0,4508	0,4528	0,4586	0,4691	0,4853	0,5096
$\overline{x}$	0,6	0,7	0,8	0,9	0,95	0,99
$\widetilde{\varphi}_-(\overline{x})$	0,5464	0,6049	0,7107	0,9634	1,3332	2,9281

Estimating the norm of the difference of the operators $\|K-\widetilde{K}\|$ and the norm of the inverse approximate operator $\|\widetilde{K}^{-1}\|$ when $p=5/4$ we have $\|K-\widetilde{K}\|<0,0183$ and $\|\widetilde{K}^{-1}\|<0,5309$ and consequently the error of the approximate solution $\widetilde{\varphi}_-(x)$ is defined by the inequality

$$\|\varphi-\widetilde{\varphi}\|_{L_{5/4}}<0,0116$$

REFERENCES

[1] Yu.Cherskii. The reduction of periodic problems of mathematical physics to singular integral equations with Cauchy's kernel. okl. Akad. Nauk SSSR, 140 (1), 69-72 (1961).

[2] V.Gavdzinski, M.G.El-Sheikh, M.E.Khalifa. The unbounded normal stress of a contact problem for a cylinder. Math. and Computers in Simulation, 49,119-128 (1999).

[3] Yu.Cherskii. Two theorems on estimation of the error and some their applications. Dokl. Akad. Nauk SSSR, 150 (2), 271-274 (1963).

[4] F.Gahov. Boundary-value problems. Pergamon Press, Oxford, Addison-Wesley (1966).