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THE STATIONARY VIBRATIONS OF AN ELASTIC HALF-PLANE SUBJECTED TO A PERIODICAL SYSTEM OF PUNCHES

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Steady problems for a periodic system of punches acting on the plane were formulated in [1] and [2]. In [3] the stationary vibrations of an elastic half-plane subjected to a periodical system of punches were considered. It should be noted that all there problems are limited to the theory of elasticity: the effect of the temperature field raised as a consequence of the contact was simply neglected.

In this paper the basic dynamical problem of the theory of thermoelasticity for a periodic system of punches is considered.

In view of the statement of the problem the boundary conditions considered here are

$$v(x,0) = v_0 e^{-i\omega t} \quad \text{if} \quad x \in \Gamma_1 \tag{1}$$

$$\sigma_y(x,0) = 0 \quad \text{if} \quad x \in \Gamma_2 \tag{2}$$

$$\tau_{xy}(x,0) = 0 \quad \text{if} \quad x \in [-\pi,\pi], \tag{3}$$

where $\Gamma_1 = [-a,a]$ and $\Gamma_2 = [-\pi,\pi] \setminus [-a,a]$.

As a requirement of the solution of this problem, the normal contact stress σ_y as well as the amplitude v_0 of the vibrations are to be found. The substitution for the component of the displacement by the expressions

$$u = \frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial y}, \qquad v = \frac{\partial\varphi}{\partial y} - \frac{\partial\psi}{\partial x} \tag{4}$$

where φ and ψ are the longitudinal and transversal respectively, into the familiar equations, [4],

$$\nabla^2 u+\frac{1}{1-2\nu}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\frac{\rho}{G}\frac{\partial^2 u}{\partial t^2}=\frac{2(1+\nu)}{1-2\nu}\alpha_T\frac{\partial T}{\partial x}\quad (x,u \underset{\leftarrow}{\rightarrow} y,v) \tag{5}$$

leads to the equations

$$\nabla^2\varphi=\frac{1}{c_1^2}\frac{\partial^2\varphi}{\partial t^2}+mT, \qquad \nabla^2\psi=\frac{1}{c_2^2}\frac{\partial^2\psi}{\partial t^2} \tag{6}$$

where $m=\frac{1+\nu}{1-\nu}\alpha_T$; α_T is the coefficient of thermal expansion, G the modulus of elasticity, ν -Poisson's ratio, $c_1^2=\frac{2G}{\rho(1-\nu)}$, $c_2^2=\frac{G}{\rho}$ and ρ is the density. We further assume that the solutions of (6) satisfy the conditions of absorption at infinity [5]. These solutions can be thought of in the form

$$\varphi(x,y,t)=e^{-i\omega t}\varphi^*(x,y) \qquad \psi(x,y,t)=e^{-i\omega t}\psi^*(x,y) \tag{7}$$

where

$$\nabla^2\varphi^*+\alpha^2\varphi^*=mT, \qquad \nabla^2\psi^*+\beta^2\psi^*=0, \tag{8}$$

$$\alpha=\frac{\omega}{c_1} \qquad \text{and} \qquad \beta=\frac{\omega}{c_2} \tag{9}$$

and consequently

$$\sigma_y^*=-2G\frac{\partial}{\partial x}\left(\frac{\partial\psi^*}{\partial y}+\frac{\partial\varphi^*}{\partial x}\right)-\rho\omega^2\varphi^*$$

$$\tau_{xy}^*=2G\frac{\partial}{\partial x}\left(\frac{\partial\varphi^*}{\partial y}-\frac{\partial\psi^*}{\partial x}\right)-\rho\omega^2\psi^* \tag{10}$$

here σ_y^*, τ_{xy}^* and in what follows v^* are related to σ_y, τ_{xy} and v by relations that are similar to (7). The boundary conditions (1) and (2) can be completed as follows

$$\overset{*}{v}(x,0) = v_-(x) + \psi_+(x), \quad \overset{*}{\sigma}_y(x,0) = \psi_-(x) \tag{11}$$

where

$$\psi_+(x) = \begin{cases} 0, & x \in \Gamma_1 \\ undetermined, & x \in \Gamma_2 \end{cases}$$
$$\psi_-(x) = \begin{cases} undetermined, & x \in \Gamma_1 \\ 0, & x \in \Gamma_2 \end{cases} \tag{12}$$
$$v_-(x) = \begin{cases} v_0, & x \in \Gamma_1 \\ 0, & x \in \Gamma_2 \end{cases}$$

According to the principle of limiting absorption, (8) are considered as limiting cases of the perturbed equations

$$\nabla^2 \varphi_\varepsilon^* + (\alpha^2 + i\omega\varepsilon)\varphi_\varepsilon^* = mT^{**}, \qquad \nabla^2 \psi_\varepsilon^* + (\beta^2 + i\omega\varepsilon)\psi_\varepsilon^* = 0 \tag{13}$$

where ε is an arbitrary small positive number.

In the same way as in [3], the corresponding hyperbolic type mixed problem is converted to a discrete Riemann problem which in turns is reduced to the singular integral equation with Hilbert kernel

$$\frac{A_{\alpha\beta}}{2\pi} \int_{-a}^{a} \cot \frac{(x-\xi)}{2} \psi_-(\xi) d\xi = 2\sum_{n=1}^{\infty} \Gamma_n^{\alpha\beta} \Psi_{n-} \sin nx - f(x) \tag{14}$$

where

$$f(x) = -if'_{\alpha\beta}(x) - v'_-(x) \tag{15}$$

The Hilbert-type integral equation (14) can be inverted in the class of integrable functions with the result

$$\psi_-(x) = \frac{1}{A_{\alpha\beta} X(x)} \left[m(x) - 2\sum_{n=1}^{\infty} \Gamma_n^{\alpha\beta} V_n(x) \Psi_{n-} + a_0 \cos \frac{x}{2} \right] \tag{16}$$

where $X(x)=\sqrt{2(\cos x-\cos a)}$,

$$V_n(x)=\frac{1}{2\pi}\int_{-a}^{a}\frac{X(\xi)\sin n\xi}{\sin\frac{\xi-x}{2}}d\xi,\quad m(x)=\frac{1}{2\pi}\int_{-a}^{a}\frac{X(\xi)f(\xi)}{\sin\frac{\xi-x}{2}}d\xi \tag{17}$$

The explicit values of the integrals $V_n(x)$ are

$$V_n(x)=\sum_{m=0}^{n}\mu_{n-m}(\cos a)\left(\cos\left(m+\frac{1}{2}\right)x\right), \tag{18}$$

$$\mu_0(\cos a)=1,\quad \mu_1(\cos a)=-\cos a,$$

$$\mu_k(\cos a)=\frac{P_{k-2}(\cos a)-P_k(\cos a)}{2k-1}$$

$(k=2,3,\dots)$

The functions $P_n(\cos a)$ are the Legendre polynomials which can be defined by the formula

$$P_n(\cos a)=\frac{1}{\pi}\int_{-a}^{a}\frac{\cos\left(n+\frac{1}{2}\right)xds}{X(x)} \tag{19}$$

The application of the Fourier transform to (16) leads to the following system of linear algebraic equations

$$A_{\alpha\beta}\Psi_{n-}+2\sum_{k=1}^{\infty}\Gamma_k^{\alpha\beta}N_{nk}\Psi_{n-}=M_n+a_0R_n,\qquad (n\in N^+), \tag{20}$$

where

$$N_{nk}=\frac{1}{2\pi}\int_{-a}^{a}\frac{V_k(x)\cos nx}{X(x)}dx,\quad M_n=\frac{1}{2\pi}\int_{-a}^{a}\frac{m(x)\cos nx}{X(x)}dx, \tag{21}$$

$$R_n = \frac{1}{2\pi} \int_{-a}^{a} \frac{\cos nx \cos \frac{x}{2}}{X(x)} dx$$

In view of (18) and (19) these coefficients can simply be found

$$N_{nk} = \frac{1}{4} \sum_{m=0}^{k} \mu_{k-m}(\cos a)\left[P_{m-n}(\cos a) + P_{m+n}(\cos a)\right] \quad (22)$$

$$R_n = \frac{1}{4}\left[P_n(\cos a) + P_{n+1}(\cos a)\right]$$

Since system (20) can in general be solved only approximately, namely using the method of truncation, we set up function spaces and sequence spaces. If $f(x) \in L_\tau[-a,a]$, $r > \frac{4}{3}$, then $\psi_-(x) \in L_p[-a,a]$, where $1 < p < \frac{4}{3}$.

Consequently the Fourier coefficients $\Psi_{n-}(x)$ will belong to ℓ_p, where $p = \frac{\rho}{\rho - 1}$. Thus we will work in the space ℓ_p $(p > 4)$ with the norm

$$\|\psi\|_{\ell_p} = \left(\sum_{n=0}^{\infty} |\psi_{n-}|^p \right)^{1/p} \quad (23)$$

where $\Psi = \{\psi_{n-}\}_{n=\overline{0,\infty}}$. The justification of truncation system (20) is a simple consequence of the following theorem whose proof of similar to that given in [6] for the case $p = 2$.

Theorem.

Suppose that

i) The homogeneous system corresponding to system (20) has only trivial solution in ℓ_p.

ii) $$\sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}\left|\Gamma_k^{\alpha\beta}N_{nk}\right|^{p/(p-1)}\right)^{p-1}<\infty$$

iii) $$\sum_{n=0}^{\infty}\left|R_n\right|^p<\infty \quad \text{and} \quad \sum_{n=0}^{\infty}\left|M_n\right|^p<\infty,$$

then the infinite system (20) has a unique solution in ℓ_p. The truncated system will also have a unique solution and the following estimate holds

$$\left\|\Psi-\Psi^N\right\|_{\ell_p}\leq Q_1\left[\sum_{n=N+1}^{\infty}\left(\sum_{k=1}^{\infty}\left|\Gamma_k^{\alpha\beta}N_{nk}\right|^{p/(p-1)}\right)p-1\right]^{1/p}+ \tag{24}$$

$$+Q_2\left[\frac{\sum_{n=N+1}^{\infty}\left|R_n\right|^p}{\sum_{n=0}^{\infty}\left|R_n\right|^p}\right]^{1/p}$$

where Q_1 and Q_2 are constants.

We shall assume that the frequency ω differs from those values for which the homogeneous system corresponding to (20) has nontrivial solutions. The fulfilment of the second and the first part of the third conditions follow from the formula

$$N_{nk}=-\frac{1}{2}\frac{n+1}{n-k}\left[P_n(\cos a)P_{k+1}(\cos a)-P_{n+1}(\cos a)P_k(\cos a)\right], \tag{25}$$

$k\geq 1$, $n\neq k$,

together with the estimate

$$\left|P_n(\cos a)\right|\leq\left(\frac{2}{\pi}\right)^{1/2}\frac{1}{\sqrt{n\sin a}}, \quad (0<a<\pi;n=1,2,\ldots) \tag{26}$$

Relations (25) can be completed according to (22) to include the case $n=k$. However we just aim here to estimate the summations in the right-hand side of (24). The consideration of the infinite parts $n > k$ on $n < k$ is sufficient for that purpose. We have

$$|N_{nk}| \sim \frac{c}{\sqrt{n}}, \qquad |N_{nk}| \sim \frac{c}{k^{3/2}} \quad \text{and} \quad R_n \sim \frac{c}{\sqrt{n}} \tag{27}$$

Therefore, conditions (ii) and the first condition of (iii) are satisfied as $p > 4$. Recall the $\Gamma_k^{\alpha\beta} = o(k^{-2})$. The satisfaction of the second part of (iii) follows because the coefficients M_n from its very definition depend only on the temperature since $V'(x) = 0$. The temperature function is the solutions of the heat equation subjected to a homogeneous Neumann condition in Γ_1 while its restriction on Γ_1 is of the form $T(x,0,t) = e^{-i\omega t} f(x)$. Thus it has the form

$$T(x,y,t) = e^{-i\omega t}\left(T_{y0}^*(0)\lambda_0 e^{-\lambda_0 y} + 2\sum_{n=1}^{\infty}\lambda_n T_{yn}^*(0) e^{inx-\lambda_n y}\right.$$

where $\lambda_n = \left(n^2 - \dfrac{i\omega}{\alpha_T}\right)^{-1/2} = o(n^{-1})$, α_T is the thermal diffusibility and the Fourier coefficients $T_{yn}(0)$ of the Neumann data are to be determine so that the above mentioned Dirichlet condition is satisfied. This determination can be achieved using the same procedures followed in this paper. The left hand side of the resulting algebraic system is similar to system (20). The distinction being only in the definition of the coefficients $\Gamma_k^{\alpha\beta}$. However the corresponding coefficients are still of order k^{-2}. Thus $T_{yn}(0) \in \ell_p$, $p > 4$. With the aid of this result together with the expressions

$$T_n^*(0) \sim \lambda_n T_{yn}^* \quad \text{and} \quad T_n^{*'}(0) \sim n\lambda_n T_{yn}^*(0)$$

we come to the conclusion that M_n is subjected to condition (iii). Finally, we have

$$\left\| \Psi - \Psi^N \right\|_{\ell_p} \le \frac{c}{(N+1)^{(p-2)/(2p)}} \cdot \tag{28}$$

Thus the approximate solution of the singular integral equation (14) is given by

$$\psi_-(x) = \frac{1}{A_{\alpha\beta} X(x)} \left[m(x) - 2\sum_{n=1}^{N} \Gamma_n^{\alpha\beta} V_n(x) \Psi_{n-} + a_0 \cos\frac{x}{2} \right] = -\sigma_y^*(x,0) \tag{29}$$

where Ψ_{n-} are the solution of system (20) truncated at the N^{th} order. The quantity a_0 included in (29) is still to be defined. In fact the equation of motion of the punch is

$$M\frac{d^2 v}{dt^2} = e^{-i\omega t}(P_0 - P_R), \tag{30}$$

where M is the mass of the punch and P_R is the reaction of the elastic half-plane:

$$P_R = -\int_{-a}^{a} \sigma_y^*(x,0)dx = \int_{-a}^{a} \psi_-(x)dx = -\frac{\pi}{A_{\alpha\beta}} a_0 \tag{31}$$

Substituting this expression together with $v = v_0 \vec{e}^{\,i\omega t}$ into (30) we get

$$a_0 = -\frac{M\omega^2 v_0 + P_0}{A_{\alpha\beta}}. \tag{32}$$

Substituting (32) into (1), when $x = 0$, we obtain the value

of the amplitude v_0.

The real values ω for which $v^*(x,0) \to \infty$, the resonance frequencies, are clearly the real roots of the resonance equations [3]

$$D_n^{\alpha\beta} = 0, \qquad \max(\alpha,\beta) \le n \le N \cdot \tag{33}$$

If $\max(\alpha,\beta)<1$, then $m=1,2,...,N$. For $n=1$, the resonance equations assume the form

$$(2G-\rho\omega^2)^2-4G^2\sqrt{1-\alpha^2}\sqrt{1-\beta^2}=0, \tag{34}$$

or in the dimensionless form

$$(2-\omega_*^2)^2-4\sqrt{1-\omega_*^2}\sqrt{1-\omega_*^2c^2}=0, \tag{35}$$

where

$$\omega_*=\frac{\omega}{c_2}, \qquad c=\frac{c_2}{c_1}, \tag{36}$$

Equation (35), coincides with Rayleigh equation for the half-space [5]. Thus, the real roots of (34) are connected with the velocity of propagation of surface Rayleigh waves.

Finally, some results of a numerical experiment are given to reveal some idea about the usefulness of the method. Let the force acting on the punch be $P_n\cos\omega t$; then

$$p(x,t)=-\mathrm{Re}(e^{-i\omega t}\psi_-(x)), \tag{37}$$

where ψ_- is given by (29). In the case

$$\overline{\omega}=0.1, \qquad N=25 \quad \text{and} \quad \nu=0.3 \tag{38}$$

the values of the contact stress $\dfrac{p(\vec{x},\tau)}{P_0}$, where $\tau=\dfrac{tc_2^3}{\alpha_T}$ is the dimensionless time, $\overline{\omega}=\dfrac{\omega_*\alpha_T}{c_2}$ is the dimensionless frequency and $\overline{x}=\dfrac{x}{a}$, corresponding to different values of the dimensionless mass

$$\overline{M}=\frac{M}{\rho\alpha_T^2} \tag{39}$$

are given in table 1 at $\tau = 2\pi$. In table 2 the values of the contact stress corresponding to different values of τ are exhibited when $\overline{M} = 1$. If $p = 5$ then the estimation of the error is subjected to the inequality

$$\left\|\Psi - \Psi^{N}\right\|_{\ell_p} \leq \frac{c}{2.6576} \quad (40)$$

Although the upper bound of the error still seems so far from its value that insures precision of the contact stress, the values shown in tables 1 and remain stable to the first three decimals when N increases beyond the 25th order.

$\overline{x} \setminus \overline{M}$	0.5	1	2	5
0.0	0.3329	0.3332	0.3337	0.3357
0.2	0.3384	0.3388	0.3393	0.3413
0.6	0.4034	0.4039	0.4045	0.4069
0.9	0.7107	0.7114	0.7125	0.7168
0.95	0.9838	0.9847	0.9863	0.9921
0.99	2.1600	2.1623	2.1657	2.1786

Table 1: The instantaneous contact stress at different values of the mass of the punch.

$\overline{x} \setminus \overline{\tau}$	1	5	10	25
0.0	0.3315	0.2927	0.1800	-0.2669
0.2	0.3771	0.2976	0.1831	-0.2714
0.6	0.4019	0.3548	0.2182	-0.3235
0.9	0.7078	0.6250	0.3844	-0.5698
0.95	0.9798	0.8651	0.5320	-0.7887
0.99	2.1515	1.8996	1.1683	-1.7320

Table 2: The time evolution of the contact stress.

In view of table 1 and 2, the values of contact stresses increase unboundedly at the vicinities of the end points while table 2 exhibiting the time evolution of the contact stress reveals that compressive stresses become tinsel stresses in the course of time. This manifests the reflection of compression waves from the corner points as tension waves.

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