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THE THERMAL FIELD FOR THIN PLATE SURROUNDED BY A FLUID

Gavdzinski V.N. [1], **El-Sheikh M.** [2], **Maltseva E.V.** [3]

[1] *Odessa State Academy of Building Engineering and Architecture, Ukraine*
[2] *Ain Shams University, Cairo, Egypt*
[3] *Odessa National Economic University, Ukraine*

Assume that the thin circular plate has an inner radius, other radius and thickness R_1, R_2 and 2σ respectively. The temperature of the fluid surrounding the surfaces $z = \pm\sigma$ is an arbitrary function depending on the time, that is $T_0(t)$. The temperatures of the fluid surrounding the circular concentric surfaces $r = R_1$ and $r = R_2$ are $T_1(t)$ and $T_2(t)$.

In view of the statement of the problem we take the equation of heat flow in the form [1]

$$\frac{\partial T}{\partial F_0} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} - B_i (T - T_0) \quad (1 = R_1 < r < R_2 = l), \qquad (1)$$

where $F_0 = \dfrac{a_T t}{\delta^2}$ $B_i = \dfrac{\alpha\delta}{\lambda}$; a_T is the coefficient of thermal diffusion, α is the of heat transfer, λ is the coefficient of thermal conductivity; B_i is the Biot’s criterion.

The boundary and initial conditions will be

$$\begin{aligned} &\frac{\partial T}{\partial r} - B_{i_1}(T - T_1) = 0 \quad \text{if} \quad r = 1, \\ &\frac{\partial T}{\partial r} + B_{i_2}(T - T_2) = 0 \quad \text{if} \quad r = l, \end{aligned} \qquad (2)$$

$$T = 0 \quad \text{if} \quad F_0 = 0 \qquad (3)$$

Now we try to seek the solution of the problem (1)-(3) in the form $T(r, F_0) = \psi(r, F_0) + \Theta(r, F_0)$, where

$$\psi(r, F_0) = (T_2(F_0) - T_1(F_0))\ (A \ln r + B) + T_1(F_0),$$

$$A = B_{i_1} B, \ \ B = B_{i_2}\left(\frac{B_{i_1}}{l} + B_{i_1} B_{i_2} \ln l + B_{i_2}\right)^{-1}.$$

The function $\Theta(r,\ F_0)$ satisfies the partial differential equation

$$\frac{\partial\Theta}{\partial F_0}=\frac{\partial^2\Theta}{\partial r^2}+\frac{1}{r}\frac{\partial\Theta}{\partial r}+B_i\Theta-f(r,\ F_0) \tag{4}$$

with the boundary and initial conditions

$$\begin{aligned}&\frac{\partial\Theta}{\partial r}-B_{i_1}\Theta=0 \quad \text{if} \quad r=1,\\ &\frac{\partial\Theta}{\partial r}+B_{i_2}\Theta=0 \quad \text{if} \quad r=l,\end{aligned} \tag{5}$$

$$\Theta(r,\ F_0)=-\psi(r,\ F_0) \ \ if \ \ F_0=0, \tag{6}$$

where

$$f(r,\ F_0)=B_i\left(\psi(r,\ F_0)-T_0(F_0)\right)+\frac{\partial\psi(r,\ F_0)}{\partial F_0} \tag{7}$$

The solution of problem (4)-(6) can be obtained by means of the finite integral transformation [2] with the kernel $rW_n(r)$, where $W_n(r)$ satisfies the equation

$$W_n''(r)+\frac{1}{r}W_n'(r)+\gamma_n^2W_n(r)=0 \tag{8}$$

with the boundary conditions

$$W_n'(1)-B_{i_1}W_n(1)=0, \qquad W_n'(l)+B_{i_2}W_n(l)=0. \tag{9}$$

Thus the passage to the transformation is carried out by the formula

$$\overline{\Theta}_n(F_0)=\int_1^l\theta(r,\ F_0)W_n(r)rdr \tag{10}$$

The general solution of equation (8) is

$$W_n(r)=A_nJ_0(\gamma_nr)+B_nY_0(\gamma_nr), \tag{11}$$

where $J_0(\gamma_nr)$ and $Y_0(\gamma_nr)$ are Bessel functions.

Substituting (11) into (5) we get the system of linear algebraic equations for A_n and B_n. This system has nontrivial solution if the determinant $\Delta(\gamma_n)$ of this homogeneous system equals zero, that is

$$\begin{aligned}\Delta(\gamma_n)=&\left(\gamma_nJ_1(\gamma_n)+B_{i_1}J_0(\gamma_n)\right)\left(\gamma_nY_1(\gamma_nl)-B_{i_2}Y_0(\gamma_nl)\right)-\\ &-\left(\gamma_nJ_1(\gamma_nl)-B_{i_2}J_0(\gamma_nl)\right)\left(\gamma_nY_1(\gamma_n)-B_{i_1}Y_0(\gamma_n)\right)=0\end{aligned} \tag{12}$$

The solution of this transcendental equation (to get the eigenvalues γ_n), may be obtained for specific values of B_i and l numerically. Here we try to find analytically an asymptotic representation for $W_n(r)$ for large values of γ_n.

To do this we make the substitution

$$\sqrt{r}W_n(r)=V_n(r), \quad x=\frac{r-1}{l-1}\pi. \tag{15}$$

Thus the boundary value problem (8)-(9) is reduced to the following problem

$$V_n''(x)+\left(\lambda_n^2+Q(x)\right)V_n(x)=0, \tag{16}$$

and

$$\begin{aligned} h_1V_n'(0)+h_2V_n(0)=0,\\ k_1V_n'(\pi)+k_2V_n(\pi)=0, \end{aligned} \tag{17}$$

where $Q(x)=\dfrac{1}{4\left(x+\dfrac{\pi}{l-1}\right)^2}, \quad h_1=k_1=\dfrac{\pi}{l-1},$

$$h_2=-\left(\frac{1}{2}+B_{i_1}\right), \quad k_2=-\left(\frac{1}{2l}-B_{i_2}\right), \quad \lambda_n=\frac{l-1}{\pi}\gamma_n.$$

Using the method explain in [3] we get the solution of Sturm-Liouvill problem (16)-(17) in the form

$$\begin{aligned} V_n(x)&=\sin\alpha\cos\lambda_n x+\lambda_n^{-1}\cos\alpha\sin\lambda_n x+\\ &+\sum_{k=1}^{\infty}\lambda_n^{-k}\int_0^x A_{kn}(x,\ y)\left(\sin\alpha\cos\lambda_n y+\lambda_n^{-1}\cos\alpha\sin\lambda_n y\right)dy \end{aligned} \tag{18}$$

where $A_{kn}(x,\ y)$ are successive iterates of the nucleus

$$A_{kn}(x,\ y)=\sin\lambda_n(x-y)Q_y; \tag{19}$$

α and β are angles defined by the equations

$$\tan\alpha=-h_1/h_2, \quad \tan\beta=k_1/k_2. \tag{20}$$

The formula (18) immediately shows that

$$V_n(x)=\sin\alpha\cos\lambda_n x+O\left(\lambda_n^{-1}\right) \tag{21}$$

which is the zero approximation of $V_n(x)$.

Now we can obtained an improved asymptotic representation of (21) with residual term $O\left(\lambda_n^{-2}\right)$ instead of $O\left(\lambda_n^{-1}\right)$. [4]

$$V_n(x)=\sin\alpha\cos\lambda_n x+\lambda_n^{-1}(\cos\alpha+P(x)\sin\alpha)\sin\lambda_n x+O\left(\lambda_n^{-2}\right), \tag{22}$$

where $P(x)=\dfrac{1}{2}\int_0^x Q(y)dy$.

Making use of the boundary condition at $x=\pi$ we have the asymptotic expression for the eigenvalues in the form

$$\lambda_n = n + \frac{1}{\pi n}\left(\frac{k_2}{k_1} - \frac{h_1}{h_2} + P(\pi)\right) + O\left(n^{-2}\right). \qquad (23)$$

Returning back to equation (4), multiplying it by $rW_n(r)$, integrating with respect to r from 1 to l, and using boundary conditions (5), (9) yields

$$\frac{d\overline{\Theta}_n(F_0)}{dF_0} + \left(\gamma_n^2 - B_i\right)\overline{\Theta}_n(F_0) + \overline{f}_n(F_0), \qquad (24)$$

where $\overline{f}_n(F_0) = \int_1^l f(r,\ F_0)W_n(r)rdr$.

The initial condition for $\overline{\Theta}_n(F_0)$ becomes

$$\overline{\Theta}_n(0) = -\overline{\psi}_n(0) \text{ with } \overline{\psi}_n(F_0) = \int_1^l \psi(r,\ F_0)W_n(r)rdr\ . \qquad (25)$$

The solution of equation (24) which satisfies initial condition (25) is

$$\overline{\Theta}_n(F_0) = \overline{\Theta}_n(0)e^{-\left(\gamma_n^2 - B_i\right)F_o} - \int_0^{F_0} \overline{f}_n(F_0)e^{\left(\gamma_n^2 - B_i\right)(t - F_0)}dt\ , \qquad (26)$$

where $\overline{\Theta}_n(0) = -\overline{\psi}_n(0) =$

$$\begin{aligned} &= -A\left(T_2(0) - T_1(0)\right)\left(\frac{A_n}{\gamma_n}\left(l\ J_1(\gamma_n l)\ln l + \frac{1}{\gamma_n}J_0(\gamma_n l) - \frac{1}{\gamma_n}J_0(\gamma_n)\right) + \right.\\ &\left. + \frac{1}{\gamma_n}\left(l\ Y_1(\gamma_n l)\ln l + \frac{1}{\gamma_n}Y_0(\gamma_n l) - \frac{1}{\gamma_n}Y_0(\gamma_n)\right)\right) - \\ &- \left(B\left(T_2(0) - T_1(0)\right) - T_1(0)\right)\left(\frac{A_n}{\gamma_n}\left(l\ J_1(\gamma_n l) - J_1(\gamma_n)\right) + \right.\\ &\left. + \frac{1}{\gamma_n}\left(l\ Y_1(\gamma_n l) - Y_1(\gamma_n)\right)\right), \end{aligned} \qquad (27)$$

where $A_n = -\dfrac{\gamma_n Y_1(\gamma_n) + B_{i_1}Y_0(\gamma_n)}{\gamma_n J_1(\gamma_n) + B_{i_1}J_0(\gamma_n)}$.

The inverse transformation (10) by virtue of the orthogonality of the eigenfunctions $W_n(r)$ with weight function r can be written in such a way

$$\Theta(r,\ F_0)=\sum_{n=1}^{\infty}\overline{\Theta}_n(F_0)\frac{W_n(r)}{N_n^2}, \tag{28}$$

where N_n is the norm of the function $W_n(r)$ defined by the formula

$$N_n^2=\int_1^l W_n^2(r)rdr. \tag{29}$$

Employing the values of the integrals [5]

$$\int J_0^2(\gamma_n r)rdr=\frac{r^2}{2}\left(J_0^2(\gamma_n r)+J_1^2(\gamma_n r)\right),$$

$$\int Y_0^2(\gamma_n r)rdr=\frac{r^2}{2}\left(Y_0^2(\gamma_n r)+Y_1^2(\gamma_n r)\right),$$

$$\int J_o(\gamma_n r)Y_0(\gamma_n r)rdr=\frac{r^2}{2}\left(J_0(\gamma_n r)Y_0(\gamma_n r)+J_1(\gamma_n r)Y_1(\gamma_n r)\right),$$

we get

$$N_n^2=\frac{1}{2}\sum_{i=1}^{2}(-1)^i\delta_i\sum_{j=0}^{1}\left(Y_j(\gamma_n\delta_i)-\frac{Y_1(\gamma_n l)}{J_1(\gamma_n l)}J_j(\gamma_n\delta_i)\right)^2 \tag{30}$$

$$(\delta_1=1,\ \delta_2=l)$$

Consequently the thermal field is determined by the formula

$$T(r,\ F_0)=(T_2(F_o)-T_1(F_0))(A\ln r+B)+T_1(F_0)+\sum_{n=1}^{\infty}\overline{\Theta}_n(F_0)\frac{W_n(r)}{N_n^2}. \tag{26}$$

Conclusion

The research results show that the exact solution of the given problem is obtained in the form of the series. Choosing the solution in the special form it was possible to improve the convergence of the series on the boundary of the plate. The received asymptotic representations of the eigenvalues and eigenfunctions allow to find the approximate numerical solution of the problem up to any prescribed accuracy.

Summary

This paper contains an exact solution for the transient temperature distribution in the thin circular plate surrounded by a fluid having the temperature $T_0(t)$ depending on time. The temperature of the plate undergoes a sudden uniform change and is steadily maintained thereafter in accordance to Newton's law of the convective heat transfer on the boundary. Using the finite integral transform with respect to the coordinate r the exact solution of the problem was obtained in the form of the infinite series.

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