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THE THERMOELASTIC CONTACT PROBLEM FOR A CYLINDER

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The contact problem of symmetric indentation of two punches in the form of circular segments, without friction, into the exterior surface of a cylinder under harmonic force $P = P_0 e^{-iwt}$ and the temperature field defined in [1] is considered.

Assuming that the radius of the cylinder in unity, in view of the statement of the problem, the boundary conditions considered here are

$$v_r(1, \theta, t) = v_0 e^{-iwt}, \qquad if \;\; \theta \in \Gamma_1 \tag{1}$$

$$\sigma_r(1, \theta, t) = 0, \qquad if \;\; \theta \in \Gamma_2 \tag{2}$$

$$\tau_{r\theta}(1, \theta, t) = 0, \qquad if \;\; \theta \in [-\pi, \pi], \tag{3}$$

where $\Gamma_1 = [-\alpha_0, \alpha_0] \cup [\pi - \alpha_0, \pi + \alpha_0]$ and $\Gamma_2 = [-\pi, \pi] / \Gamma_1$. In addition, the stresses σ_r and $\tau_{r\theta}$ as well as the displacement v_r, are bounded as $r \to 0$.

As a requirement of the solution of this problem, the normal contact stress σ_r as well as the amplitude v_0 of the vibration are to be found. The substitution for the components of the displacement by the expressions:

$$v_r = \frac{\partial \Phi}{\partial r} + \frac{1}{r}\frac{\partial \Psi}{\partial \theta}, \qquad v_\theta = \frac{1}{r}\frac{\partial \Phi}{\partial \theta} - \frac{\partial \Psi}{\partial r}, \tag{4}$$

where Φ and Ψ are the wave potentials, into the equation of motion in displacements [2], leads to the equations:

$$\nabla^2 \Phi = \frac{1}{c_1^2}\frac{\partial^2 \Phi}{\partial t^2} + mT, \quad \nabla^2 \Psi = \frac{1}{c_2^2}\frac{\partial^2 \Psi}{\partial t^2} + mT, \tag{5}$$

where $m = \dfrac{1+\upsilon}{1-\upsilon}\alpha_T$; α_T is the coefficient of thermal expansion $c_1^2 = 2G / \rho(1-\upsilon)$, $c_2^2 = G/\rho$, G is the modulus of elasticity, υ is the Poisson ratio, ρ is the density.

The potentials Φ and Ψ can be thought of in the form

$$\Phi(r, \theta, t) = \Phi^*(r, \theta) e^{-iwt}, \qquad \Psi(r, \theta, t) = \Psi^*(r, \theta) e^{-iwt}, \quad (6)$$

where $\nabla^2\Phi^* + \alpha^2\Phi^* = mT^*, \quad \nabla^2\Psi^* + \beta^2\Psi = 0,$ (7)

$$\alpha = \frac{w}{c_1}, \qquad \beta = \frac{w}{c_2},$$

and consequently the substitution of (4) into the Hooke law yields

$$\frac{\sigma_r^*}{2G} = \left(\frac{\partial^2}{\partial r^2} - \frac{\upsilon}{1-2\upsilon}\alpha^2\right)\Phi^* + \left(\frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\frac{\partial \Psi^*}{\partial \theta} - mT^*, \quad (8)$$

$$\frac{\sigma_\theta^*}{2G} = -\left(\frac{\partial^2}{\partial r^2} - \frac{1-\upsilon}{1-2\upsilon}\alpha^2\right)\Phi^* - \left(\frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\frac{\partial \Psi^*}{\partial \theta} - mT^*, \quad (9)$$

$$\frac{\tau_{\theta r}^*}{2G} = \left(\frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\frac{\partial \Phi^*}{\partial \theta} - \left(\frac{\partial^2}{\partial r^2} + \frac{\beta^2}{2}\right)\Psi^*. \quad (10)$$

The boundary conditions (1) and (2) can be completed as follows:

$$v_r^*(1, \theta) = v_-(\theta) + \psi_+(\theta), \qquad \frac{\sigma_r^*(1, 0)}{2G} = \psi_-(\theta) \quad (11)$$

where

$$\psi_+(\theta) = \begin{cases} 0, & \theta \in \Gamma_1 \\ \text{undetermined}, & \theta \in \Gamma_2 \end{cases}; \qquad \psi_-(\theta) = \begin{cases} \text{undetermined}, & \theta \in \Gamma_1 \\ 0, & \theta \in \Gamma_2 \end{cases};$$

$$v_-(\theta) = \begin{cases} v_0, & \theta \in \Gamma_1 \\ 0, & \theta \in \Gamma_2 \end{cases}. \quad (12)$$

In the same way as in [3], the corresponding hyperbolic type mixed problem is converted to a discrete Riemann problem which in turn is reduced to the singular integral equation with Hilbert kernel

$$\frac{A_{\alpha\beta}}{\pi}\int_{-\alpha_0}^{\alpha_0}\cot(\xi - \theta)\psi_-(\xi)d\xi = -2\sum_{n=1}^{\infty}\Gamma_{2n}^{\alpha\beta}\Psi_{2n-}\sin 2n\theta + f'_{\alpha\beta}(\theta), \quad (13)$$

where

$$f'_{\alpha\beta}(\theta) = AB_{\alpha\beta}\left[\frac{\partial T^*(1,\theta)}{\partial\theta} - \frac{\partial^2 T^*(1,\theta)}{\partial r\,\partial\theta}\right] - 4i\sum_{n=1}^{\infty}\left[T^*_{2n}(1) - T^{*\,\prime}_{2n}(1)\right] nK^{\alpha\beta}_{2n}\cos 2n\theta$$

$$+ 2i\sum_{n=1}^{\infty}\left[\left(m + \frac{A\alpha^2\upsilon}{1-\upsilon}\right)T^*_{2n}(1) - AT^{*\,\prime\prime}_{2n}(1)\right]\Gamma^{\alpha\beta}_{2n}\cos 2n\theta - A\frac{\partial^2 T^*(1,0)}{\partial r\,\partial\theta} -$$

$$-\frac{A_{\alpha\beta}i}{\pi}\int_{-\alpha_0}^{\alpha_0}\left[\left(m + \frac{A\alpha^2\upsilon}{1-2\upsilon}\right)T^*(1,\xi) - A\frac{\partial^2 T^*(1,\xi)}{\partial r^2}\right]\cot(\xi-\theta)\,d\xi,$$

where

$$A = \frac{m}{\alpha^2 - \frac{iw}{a_r}}, \quad A_{\alpha\beta} = \frac{\beta^2}{2\left[\frac{\beta^2}{2}(e-1) - \alpha^2\left(\frac{1-\upsilon}{1-2\upsilon} - \frac{e}{2}\right)\right]},$$

$$B_{\alpha\beta} = \frac{\alpha^2\left(\frac{1-\upsilon}{1-2\upsilon} - \frac{e}{2}\right) - \frac{\beta^2}{2}e}{2\left[\frac{\beta^2}{2}(e-1) - \alpha^2\left(\frac{1-\upsilon}{1-2\upsilon} - \frac{e}{2}\right)\right]}, \qquad \Gamma^{\alpha\beta}_n = A_{\alpha\beta}\,sgn\left(n + \frac{1}{2}\right) - nX_n,$$

e is the Euler number;

$K^{\alpha\beta}_n = B_{\alpha\beta} - n^2\Omega_n$, and additionally $\Gamma^{\alpha\beta}_n = K^{\alpha\beta}_n = O\left(\frac{1}{n^2}\right)(n \to \infty)$.

The Hilbert-type integral equation (13) can be inverted in the class of integrable functions [4], with the result

$$\psi_-(\theta) = \frac{1}{A_{\alpha\beta}X(\theta)}\left[m_{\alpha\beta}(\theta) + 2\sum_{n=1}^{\infty}\Gamma^{\alpha\beta}_{2n}V_{2n}(\theta)\Psi_{2n-} + a_0\cos\theta\right], \quad (14)$$

where

$$m_{\alpha\beta}(\theta) = -\frac{1}{\pi}\int_{-\alpha_0}^{\alpha_0}\frac{X(\xi)f'_{\alpha\beta}(\xi)}{\sin(\xi-\theta)}d\xi, \qquad V_{2n}(\theta) = \frac{1}{\pi}\int_{-\alpha_0}^{\alpha_0}\frac{X(\xi)\sin 2n\xi}{\sin(\xi-\theta)}d\xi,$$

$$X(\theta) = \sqrt{2(\cos 2\theta - \cos 2\alpha_0)}.$$

The application of the finite Fourier transform to (14) leads to the following infinite system of linear algebraic equations:

$$A_{\alpha\beta}\Psi_{2n-} = 2\sum_{k=1}^{\infty}\Gamma^{\alpha\beta}_{2k}N_{nk}\Psi_{2n-} + M^{\alpha\beta}_{2n} + a_0R_n \quad (n \in N^+), \quad (15)$$

where

$$N_{nk}=\frac{1}{\pi}\int_{-\alpha_0}^{\alpha_0}\frac{V_{2k}(\theta)cos\,2n\theta}{X(\theta)}d\theta, \qquad M_{2n}^{\alpha\beta}=\frac{1}{\pi}\int_{-\alpha_0}^{\alpha_0}\frac{m_{\alpha\beta}(\theta)cos\,2n\theta}{X(\theta)}d\theta,$$

$$\mathrm{R_n}=\frac{1}{\pi}\int_{-\alpha_0}^{\alpha_0}\frac{\cos 2n\theta\cos\theta}{\mathrm{X}(\theta)}\mathrm{d}\theta.$$

In view of [3] coefficients N_{nk} and R_n can easily be found

$$N_{nk}=\frac{1}{4}\sum_{m=0}^{k}\mu_{k-m}(\cos 2\alpha_0)[P_{m-n}(\cos 2\alpha_0)+P_{m+n}(\cos 2\alpha_0)], \tag{16}$$

$$R_n=\frac{1}{4}[P_n(\cos 2\alpha_0)+P_{n-1}(\cos 2\alpha_0)],$$

where $\mu_0(\cos\alpha_0)=1,\ \mu_1(\cos\alpha_0)=-\cos\alpha_0,$

$$\mu_k(\cos\alpha_0)=\frac{P_{k-2}(\cos\alpha_0)-P_k(\cos\alpha_0)}{2k-1},\ (k=2,3,\ldots).$$

The functions $P_n(\cos\alpha_0)$ are the Legendre polynomials which can be defined by the formula:

$$P_n(\cos\alpha_0)=\frac{1}{\pi}\int_{-\alpha_0}^{\alpha_0}\frac{\cos\left(n+\frac{1}{2}\right)\theta}{X(\theta)}d\theta. \tag{17}$$

Since system (15) can in general be solved approximately, namely using the method of truncation, we set up function spaces and sequence spaces. The solutions (14) of equation (13) is in $L_\rho(-\alpha_0,\alpha_0)$, where $1<\rho<\frac{4}{3}$ [5]. Consequently the Fourier coefficients Ψ_{n-} will belong to l_p where $p=\rho/(\rho-1)$ [5]. Thus we will work in the space $l_p\ (p>4)$ with the norm

$$\|\Psi\|_{lp}=\left(\sum_{n=0}^{\infty}|\Psi_{n-}|^p\right)^{1/p} \tag{18}$$

where $\Psi=\{\Psi_{n-}\}_{n=\overline{0,\infty}}$. The justification of truncating system (15) is a simple consequence of the following theorem whose proof is similar to that given in [6] for the case $p=2$

Theorem. Suppose that:

1. the homogeneous system corresponding to system (15) has only trivial solution in l_p;

2. $\sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}\left|\Gamma_{2k}^{\alpha\beta}N_{nk}\right|^{p/(p-1)}\right)^{p-1}<\infty$;

3. $\sum_{n=0}^{\infty}\left|R_n\right|^p<\infty$.

Than the infinite system (15) has a unique solution in l_p. The truncated system will also have a unique solution and the following estimate holds:

$$\left\|\Psi-\Psi^N\right\|_{l_p}\le Q_1\left[\sum_{n=N+1}^{\infty}\left(\sum_{k=1}^{\infty}\left|\Gamma_{2k}^{\alpha\beta}N_{nk}\right|^{p/(p-1)}\right)^{p-1}\right]^{1/p}+$$

$$+Q_2\left[\frac{\sum_{n=N+1}^{\infty}\left|R_n\right|^p}{\sum_{n=0}^{\infty}\left|R_n\right|^p}\right]^{1/p} \qquad (19)$$

where Q_1 and Q_2 are constants.

We shall assume that the frequency w differs from those values for which the homogeneous system corresponding to (15) has nontrivial solutions. The fulfillment of the second and the third conditions follows from (16) and (17) for $n=k$ together with the formula:

$$N_{nk}=-\frac{n+1}{2(n-k)}\left[P_n(\cos 2\alpha_0)P_{k+1}(\cos 2\alpha_0)-,\right.$$
$$\left.-P_{n+1}(\cos 2\alpha_0)P_k(\cos 2\alpha_0)\right],\quad k\ge 1,\ \ n\ne k \qquad (20)$$

with the estimate

$$\left|P_m(\cos\alpha_0)\right|\le\left(\frac{2}{\pi}\right)^{1/2}\frac{1}{\sqrt{n}\sin\alpha_0}\ \left(0<a_0<\pi,\ n=1,\ 2,\ ...\right) \qquad (21)$$

Thus, we have

$$\left|N_{nk}\right|\sim\frac{c}{\sqrt{nk^3}}\quad\text{and}\quad R_n\sim\frac{c}{\sqrt{n}} \qquad (21)$$

Therefore, conditions (2) and (3) are satisfied as $p > 4$. Recall that $\Gamma_k^{\alpha\beta} \sim O\left(k^{-2}\right)$. Additionally, we have

$$\left\| \Psi - \Psi^M \right\|_{l_p} \leq \frac{c}{\left(\mathrm{N}+1\right)^{(p-2)/2p}} \tag{22}$$

Thus the approximate solution of the singular integral equation (13) is given by:

$$\psi_-(\theta) = \frac{1}{A_{\alpha\beta} X(\theta)} \left[m_{\alpha\beta}(\theta) + 2\sum_{n=1}^{N} \Gamma_{2n}^{\alpha\beta} V_{2n}(\theta) \Psi_{2n-} + a_0 \cos\theta \right] \tag{23}$$

where Ψ_{2n-} are the solutions of system (15) truncated at the Nth order.

The equivalence condition for the approximate solution (23) can be written in the form:

$$-A_0^* I_1(\alpha) + 2\sum_{n=1}^{N} A_{2n}^* I_{2n}'(\alpha) + 2in\, B_{2n}^* I_{2n}(\beta) + A \frac{\partial T^*(1,\, 0)}{\partial r} = v_0 \tag{24}$$

The quantity a_0 included in (23) is still to be defined. In fact the equation of motion of the punch is [7]

$$M \frac{d^2 V_r}{dt^2} = e^{-iwt}\left(P_0 - P_R\right) \tag{25}$$

where M is the mass of the punch. P_0 the amplitude of the force acting on the punch, and P_R the reaction of the elastic cylinder:

$$P_R = -\int_{-\alpha_0}^{\alpha_0} \sigma_r(1,\theta) d\theta = -2G \int_{-\alpha_0}^{\alpha_0} \psi_-(\theta) d\theta = -\frac{2G\, a_0\, \pi}{A_{\alpha\beta}}.$$

Substituting the expression $v_r = v_0 e^{-iwt}$ into (25) we have

$$-Mw^2 v_0 = P_0 + \frac{2G\, a_0\, \pi}{A_{\alpha\beta}} \tag{26}$$

Thus, the amplitude v_0 and the quantity a_0 can be calculated from equations (24) and (26).

Using formulas (11), (23) we get the expression for the contact stress

$$p(r,\ t) = -\sigma_r(1,\ \theta,\ t) = -\frac{2Ge^{-iwt}}{A_{\alpha\beta}\sqrt{2(\cos 2\theta - \cos 2\alpha_0)}} \cdot$$
$$\cdot\left[m_{\alpha\beta}(\theta) + 2\sum_{n=1}^{N}\Gamma_{2n}^{\alpha\beta} V_{2n}(\theta)\Psi_{2n-} + a_0 \cos\theta\right] \tag{27}$$

The real values w for which $v_r(1,\ \theta \to \infty)$, the resonance frequencies, are the real roots of the resonance equation which for $n = 0$ assumes the form:

$$I_1(\alpha) - \frac{1-\upsilon}{1-2\upsilon}\alpha I_0(\alpha) = 0 \tag{28}$$

where I_0 and I_1 are Bessel functions of he first kind. On supposing that $\alpha = \frac{w}{c_1}$ is large and making use of the asymptotic formulas:

$$I_0(\alpha) = \sqrt{\frac{2}{\pi\alpha}}\cos\left(\alpha - \frac{\pi}{4}\right) + O\left(\frac{1}{\alpha}\right), \quad I_1(\alpha) = \sqrt{\frac{2}{\pi\alpha}}\sin\left(\alpha - \frac{\pi}{4}\right) + O\left(\frac{1}{\alpha}\right),$$

the roots of equation (28) are $\alpha = \frac{3\pi}{4} + k\pi$ or $w = c_1\left(\frac{3\pi}{4} + k\pi\right)$.

Let $\overline{w} = \frac{w}{c_2^2} = 0{,}1,\quad N = 25,\ \upsilon = 0{,}3$ and $M = 1$

The values of the contact stress $p(\overline{\theta}, \tau)/P_0$ are exhibited at different values of the dimensionless coordinate $\overline{\theta} = \theta/\alpha_0$ when the dimensionless time $\tau = tc_1 = 2\pi$. If $p = 5$ then the estimation of the error is subjected to the inequality:

$$\left\|\Psi - \Psi^N\right\|_{l_p} \le \frac{c}{3{,}278}. \tag{29}$$

Although this upper bound on the error still seems far from a value which would ensure precision of the contact stress, the values shown in the table remain stable to the first three decimals when N increases beyond the 25th order:

$\overline{\theta}$	0,1	0,2	0,6	0,9	0,95	0,99
$p(\overline{\theta}, 2\pi)/P_0$	0,6371	0,6681	0,8236	1,1874	1,4871	3,1635

Note that the values of the contact stresses increase unboundedly at the vicinities of the end points of the contact interval.

CONCLUSION

On choosing the number N and using the formula (27) we can get an approximate solution of the problem to find a contact stress up to any prescribed accuracy.

SUMMARY

The problem is formulated into a singular integral equation of Hilbert type, its solution providing an expression for the physically important unbounded normal stress. The integral equation is converted into an infinite system of algebraic equations the solution of which can be obtained by means of truncation method. The truncation is justified and the error is estimated.

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