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THE PROBLEM OF HEAT CONDUCTIVITY WITH MOVING BOUNDARY CONDITIONS

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In the half-space there is considered the equation of heat conduction with the mixed boundary conditions traveling with constant velocity α along the x-axis and β along the y-axis.

On applying the Fourier transform the problem is reduced to the Riemann boundary value problem. The investigation of this problem is carried out and the solution of the posed problem is obtained in a closed form.

We will consider the equation of heat conduction [1]

$$\frac{\partial T}{\partial t}=a^2\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right) \tag{1}$$

in the half-space $-\infty < x < +\infty$, $-\infty < y < +\infty$, $z > 0$ for $-\infty < t < \infty$; where $T(x, y, z, t)$ is the temperature and a is the coefficient of thermal conductivity.

Let us specified at $z = 0$ the mixed boundary conditions traveling with constant velocity α along the x-axis and β along the y-axis

$$T(x-\alpha t,\ y-\beta t,\ 0)= g_+(x-\alpha t,\ y-\beta t),\ x-\alpha t>0; \tag{2}$$

$$\frac{\partial T(x-\alpha t,\ y-\beta t,\ 0)}{\partial z}= h_-(x-\alpha t,\ y-\beta t),\ x-\alpha t<0. \tag{3}$$

We seek the solution $T(x, y, z, t)$ of the problem (1) – (3) bounded as $z \to \infty$.

On introducing new variables by formulas $\xi = x-\alpha t$, $\eta = y-\beta t$ we obtain

$$\alpha\frac{\partial T}{\partial x}+\beta\frac{\partial T}{\partial y}=\frac{\partial^2 T}{\partial \xi^2}+\frac{\partial^2 T}{\partial \eta^2}+\frac{\partial^2 T}{\partial z^2} \tag{4}$$

On applying the Fourier transform [2] to the equation (4) with respect to variables ξ and η we have

$$\frac{\partial^2 \overline{T}}{\partial z^2}-\left(x^2+y^2+\frac{\alpha x+\beta y}{a^2}i\right)\overline{T}=0,\quad z>0 \tag{5}$$

where

$$\overline{T}(x,y,z)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}T(\xi,\eta,z)e^{i(x\xi+y\eta)}d\xi\, d\eta=VT.$$

By symbol V we denote the operator Fourier transform.

Assuming x and y to be parameters we find the solution of the equation (5)

$$\overline{T}(x,y,z)=A(x,y)e^{-kz}+B(x,y)e^{kz},\quad z>0, \tag{6}$$

where

$$k=\left(x^2+y^2+\frac{\alpha x+\beta y}{a^2}i\right)^{\frac{1}{2}}.$$

Suppose that

$$\mathrm{Re}=\left(x^2+y^2+\frac{\alpha x+\beta y}{a^2}i\right)^{\frac{1}{2}}\geq 0. \tag{7}$$

Taking into account the boundness of the function $\overline{T}(x,y,z)$ as $z\to\infty$ and the inequality (7) we arrive at the identity $B(x,y)=0$.

The solution of the equation (5) takes the form

$$\overline{T}(x,y,z)=A(x,y)e^{-kz},\quad z>0 \tag{8}$$

Let us complete the boundary conditions (2), (3) by introducing unknown functions $f_+(\xi,\eta)$, $f_-(\xi,\eta)$ such that

$$f_+(\xi,\eta)=0 \text{ for } \xi<0; \qquad f_-(\xi,\eta)=0 \text{ for } \xi>0;$$

$$T(\xi,\eta,0)=g_+(\xi,\eta)+f_-(\xi,\eta)\quad(-\infty<\xi<+\infty), \tag{9}$$

$$\frac{\partial T(\xi,\eta 0)}{\partial z}=h_-(\xi,\eta)+f_+(\xi,\eta)\quad(-\infty<\xi<+\infty). \tag{10}$$

On employing the Fourier transform to the boundary conditions (9) – (10) we have

$$\overline{T}(x,y,0)=\overset{+}{G}(x,y)+\overset{-}{F}(x,y) \qquad (-\infty<x<+\infty) \qquad (11)$$

$$\frac{\partial\overline{T}(x,y,0)}{\partial z}=\overset{-}{H}(x,y)+\overset{+}{F}(x,y) \qquad (-\infty<x<+\infty) \qquad (12)$$

By means of (8) we get

$$\overline{T}(x,y,0)=A(x,y),$$

$$\frac{\partial\overline{T}(x,y,0)}{\partial z}=-k\,A(x,y).$$

On using these relations we find from (11) and (12)

$$\overset{-}{F}(x,y)=\frac{\overset{+}{F}(x,y)}{\left(x^2+y^2+\dfrac{\alpha x+\beta y}{a^2}i\right)^{\frac{1}{2}}}-N(x,y) \qquad (13)$$

On assuming y to be a parameter we have the following boundary value problem: to find two functions $\overset{+}{F}(x,y)$, $\overset{-}{F}(x,y)$ analytically continuable into the upper and lower semi-planes respectively if on the real axis they satisfy the linear relation (13).

Let us carry out the factorization of the coefficient of the problem (13)

$$\left(x^2+y^2+\frac{\alpha x+\beta y}{a^2}i\right)^{\frac{1}{2}}=(x-x_1)^{\frac{1}{2}}(x-x_2)^{\frac{1}{2}},$$

where $$x_{1,2}=\frac{-i\alpha\pm\left[-\alpha^2-4a^2\left(a^2y^2+i\beta y\right)\right]^{\frac{1}{2}}}{2a^2}.$$

Let us consider the discriminant

$$D=-\alpha^2-4a^4y^2-4ia^2\beta y=c+id$$

and $(c+id)^{\frac{1}{2}}=u+iv$, where

$$u^2=\frac{1}{2}\left(c+\sqrt{c^2+d^2}\right),\quad v^2=\frac{1}{2}\left(-c+\sqrt{c^2+d^2}\right).$$

We can not take the values of u and v arbitrary because the sign of the product $u\,v$ must coinside with the sign of d. This gives two possible combinations of values of u and v, that is, two numbers of the type $u+iv$ that are the values of the square root of $c+id$.

We have $d=d(y)$. Hence the sign of the product depends on the sign of y.

<u>The case</u> $y>0$. Here $d<0$, and the following two combinations of values of u and v are possible

1) $u>0$, $v<0$.

$$x_1' = \frac{-i\alpha+u+iv}{2a^2} = u' - iv',$$

where $u' = \dfrac{u}{2a^2}$, $v' = \dfrac{\alpha - v}{2a^2} > 0$, since $|\alpha|<|v|$;

$$x_2' = \frac{-i\alpha-u-iv}{2a^2} = -u' + iv'',$$

where $v'' = -\dfrac{\alpha+v}{2a^2} > 0$ since $|\alpha|<|v|$.

Hence $\left(x^2+y^2+\dfrac{\alpha x+\beta y}{a^2}i\right)^{\frac{1}{2}} = (x-x_1')^{\frac{1}{2}}(x-x_2')^{\frac{1}{2}}$,

where

$x-x_1' = x-u'+iv'$, $\quad \mathrm{Im}(x-x_1')>0$;

$x-x_2' = x+u'-iv''$, $\quad \mathrm{Im}(x-x_2')<0$.

On taking advantage of the symbols $\sqrt{z}^{+}$, $\sqrt{z}^{-}$ we obtain the required factorization

$$\left(x^2+y^2+\frac{\alpha x+\beta y}{a^2}i\right)^{\frac{1}{2}} = \sqrt{x-x_1'}^{+}\sqrt{x-x_2'}^{-} \qquad (14)$$

2) $u<0$, $v>0$.

In this case

$$x_1'' = \frac{-i\alpha+u+iv}{2a^2} = -u' + iv'',$$

$$x_2'' = \frac{-i\alpha-u+iv}{2a^2} = u' + iv',$$

where $\operatorname{Im}(x-x_1'')<0$, $\operatorname{Im}(x-x_2'')>0$.

Therefore the factorization is

$$\left(x^2+y^2+\frac{\alpha x+\beta y}{a^2}i\right)^{\frac{1}{2}}=-\sqrt{x-x_1''}^{\,-}\sqrt{x-x_2''}^{\,+} \tag{15}$$

<u>The case</u> $y<0$.

Here $d>0$ and the following two combinations of values of u and v are possible

1) $u<0$, $v<0$.

$x_1'=-u'-iv'$, $\qquad x_2'=u'+iv''$;

$$\left(x^2+y^2+\frac{\alpha x+\beta y}{a^2}i\right)^{\frac{1}{2}}=-\sqrt{x-x_1'}^{\,+}\sqrt{x-x_2'}^{\,-} \tag{16}$$

2) $u>0$, $v>0$.

$x_1''=u'+iv''$, $\qquad x_2''=-u'-iv'$;

$$\left(x^2+y^2+\frac{\alpha x+\beta y}{a^2}i\right)^{\frac{1}{2}}=-\sqrt{x-x_1''}^{\,-}\sqrt{x-x_2''}^{\,+} \tag{17}$$

The factorization has been carried out. Now we shall proceed to the solution of the problem (13)

$$F^-(x,y)\sqrt{x-x_2'}^{\,-}=\frac{F^+(x,y)}{\sqrt{x-x_1'}^{\,+}}-N(x,y)\sqrt{x-x_1'}^{\,+}\sqrt{x-x_2'}^{\,-} \tag{18}$$

On solving the jump problem [3]

$$F^-(x,y)\sqrt{x-x_2'}^{\,-}-\left[N(x,y)\sqrt{x-x_1'}^{\,+}\sqrt{x-x_2'}^{\,-}\right]^-=$$

$$=\frac{F^+(x,y)}{\sqrt{x-x_1'}^{\,+}}-\left[N(x,y)\sqrt{x-x_1'}^{\,+}\sqrt{x-x_2'}^{\,-}\right]^+=Q_n(x,y),$$

where $Q_n(x,y)=\sum_{k=0}^{n}a_k(y)x^k$,

and by symbols $[M(x)]^+$, $[M(x)]^-$ we denote

$$[M(x)]^{+}=V\left\{\frac{\operatorname{sgn}x+1}{2}V^{-1}M(x)\right\},$$

$$[M(x)]^{-}=V\left\{\frac{\operatorname{sgn}x-1}{2}V^{-1}M(x)\right\},$$

we obtain the solution of the boundary value problem (13)

$$F^{-}(x,y)=\frac{1}{\sqrt{x-x_2'}^{-}}\left\{\left[N(x,y)\sqrt{x-x_1'}^{+}\sqrt{x-x_2'}^{-}\right]^{-}+Q_n(x,y)\right\},$$

$$F^{+}(x,y)=\frac{1}{\sqrt{x-x_1'}^{+}}-\left\{\left[N(x,y)\sqrt{x-x_1'}^{+}\sqrt{x-x_2'}^{-}\right]^{+}+Q_n(x,y)\right\}. \quad (19)$$

In accordance with (11)

$$A(x,y)=G^{+}(x,y)+F^{-}(x,y)$$

and on applying the inverse Fourier transform [4] we find the solution of the posed problem in a closed form

$$T(\xi,\eta,z)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[G^{+}(x,y)+F^{-}(x,y)\right]e^{-i(x\xi+y\eta)-kz}dx\,dy \quad (20)$$

For the factorizations (15) – (17) the solution can be obtained in the similar way.

References

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