УДК 539.3

THE FIRST FUNDAMENTAL PLANE PROBLEM OF THERMOELASTICITY FOR ANISOTROPIC SOLID

Gavdzinski V.N. *(Odessa State Academy of Structure and Architecture, Odessa)*, **El-Sheikh M.** *(Ain Shams University, Cairo, Egypt)*, **Maltseva E.V.** *(Odessa State Economic University, Odessa)*

This paper presents a method for the investigation of the boundary value problem indicated below which corresponds to the first fundamental problem of the thermoelasticity. With the aid of the integral representation this problem was reduced to the singular integral equation.
The necessary and sufficient conditions for the solvability of this equation were obtained.

1. Let the anisotropic with respect to elastic and thermal properties body occupies the connected domain $\mathbf{D}$ of the plane $\mathbf{x,y}$, bounded by smooth, closed contours $\mathbf{L_1, L_2, ..., L_m, L_{m+1}}$ not intersecting one another of which $\mathbf{L_{m+1}}$ contains all the others. The positive direction of $\mathbf{L_k}$ will be such that $\mathbf{D}$ lies to the left when $\mathbf{L_k}$ is described in that direction.

Let the external stresses be given on the contours $\mathbf{L_k}$. Besides we suppose, that the solid is subjected to the steady – state temperature field $\mathbf{T(x,y)}$.

The temperature field is governed by the following equation [1]

$$\mathbf{k_{11}\frac{\partial^2 T}{\partial x^2} + 2k_{12}\frac{\partial^2 T}{\partial x\,\partial y} + k_{22}\frac{\partial^2 T}{\partial y^2} = 0}, \qquad (1)$$

where $\mathbf{k_{ij}}$ – coefficients of thermal conductivity.

The temperature on the closed contours $\mathbf{L_k}$ is thought to be given. In the presence of three planes of thermal symmetry at every point the coefficient $\mathbf{k_{12}}$ is equal to zero.

The general solution of equation (1) is expressed in terms of an arbitrary analytic function $\mathbf{F}$ of the complex variable $\mathbf{Z_3 = x + m^* y}$ by the formula

$$\mathbf{T = 2\,Re\,F(Z_3)}, \qquad (2)$$

where a $\mathbf{m^*}$ is one of the complex roots of the characteristic equation

$$k_{22}m^{*2} + 2k_{12}m^{*} + k_{11} = 0. \qquad (3)$$

The solution of this problem is reduced [2], [3] to finding three analytic functions $\phi_1(Z_1), \phi_2(Z_2), F(Z_3)$ of the complex variables $Z_1 = x + s_1 y$, $Z_2 = x + s_2 y$, $Z_2 = x + m^* y$;

s_1 and s_2 complex roots of the characteristic equation [2].

Complex variables Z_1, Z_2, Z_3 change in the domains D_1, D_2, D_3 obtained from the domain D by the corresponding affine transformation.

Denote images of L_k, $L = \sum_{k=1}^{m+1} L_k$ for these affine transformations by

$$L_k^j,\ L^j\ (j = 1,2,3)$$

On the boundary the functions $\phi_1(Z_1), \phi_2(Z_2), F(Z_3)$ satisfy the boundary conditions

$$2\,\mathrm{Re}\left[F(t_3)\right] = f(t) + g(t); \qquad (4)$$

$$2\,\mathrm{Re}\left[\phi_1(t_1) + \phi_2(t_2) + \psi(t_3)\right] = f_1(t) + g_1(t) + \sum_{k=1}^{m+1} C_k^{(1)}\beta_k(t)$$

$$2\,\mathrm{Re}\left[s_1\phi_1(t_1) + s_2\phi_2(t_2) + m^*\psi(t_3)\right] = f_2(t) + g_2(t) + \sum_{k=1}^{m+1} C_k^{(2)}\beta_k(t)$$

$$(t \in L), \qquad (5)$$

where t is a boundary point of the domain D; t_1, t_2, t_3 are corresponding boundary points of the domains D_1, D_2, D_3; $\beta_k(t)$ are harmonic functions. On the contour they assume the following values;

$$\beta_k(t) = \begin{cases} 1 \text{ for } t \in L_k \\ 0 \text{ on remaining contours.} \end{cases}$$

$f(t)$ is the temperature given on the contour L; $f_1(t), f_2(t)$ are expressed in terms of the vector of external forces (X_n, Y_n) in the following manner

$$f_1(t) = -\int_0^s Y_n(s)\,ds, \quad f_2(t) = -\int_0^s X_n(s)\,ds;$$

$C_k^{(i)}$ $(i=1,2)$ are real constants that must be defined; the functions $g(t), g_1(t), g_2(t)$, are of the form

$$g(t)=\sum_{k=1}^{m}\mathrm{Re}\left[A_k\ln\left(t_3-Z_{3,k}\right)\right],$$

$$g_1(t)=\sum_{k=1}^{m}\mathrm{Re}\left[B_k\ln\left(t_1-Z_{1,k}\right)+C_k\ln\left(t_2-Z_{2,k}\right)\right],$$

$$g_2(t)=\sum_{k=1}^{m}\mathrm{Re}\left[s_1B_k\ln\left(t_1-Z_{1,k}\right)+s_2C_k\ln\left(t_2-Z_{2,k}\right)\right],$$

$Z_{j,k}$ $(j=1,2,3)$ are points arbitrary chosen within contours L_k^1, L_k^2, L_k; real constants A_k are expressed in terms of dislocation's characteristic [3]. Complex constants B_k, C_k are linearly expressed in terms of X_k and Y_k [4]; the function $\psi(Z_3)$ is given by the formula [5].

$$\psi(Z_3)=\frac{E_o\int F(Z_3)\,dZ_3}{\left(m^*-s_1\right)\left(m^*-s_2\right)\left(m^*-\overline{s_1}\right)\left(m^*-\overline{s_2}\right)} \qquad (6)$$

where $$E_o=\frac{1}{a_{11}}\left(\alpha_6 m^*-\alpha_1 m^{*2}+\alpha_2\right).$$

In the boundary condition (4) changing t_3, to $\alpha_1(t)$ we shall have

$$2\,\mathrm{Re}\left\{F\left[\alpha_1(t)\right]\right\}=f(t)+g(t) \qquad (7)$$

On multiplying both sides of equation (7) by the real regulating multiplier $R(t)$ [6] we reduce it to the form

$$2\,\mathrm{Re}\left\{F\left[\alpha_1(t)\right]R(t)\right\}=R(t)\left[f(t)+g(t)\right]. \qquad (8)$$

Since $F\left[\alpha_1(Z)\right]R(Z)$ is an analytic function then it is possible to define it by means of Shwarts operator according to the formula

$$F\left[\alpha_1(Z)\right]R(Z)=\frac{1}{4\pi}\int_L\frac{\partial M(Z,\tau)}{\partial n}\mathrm{Re}\left\{F\left[\alpha_1(\sigma)\right]R(\sigma)\right\}d\sigma, \qquad (9)$$

where $\tau = \tau(\sigma)$ is the complex coordinate of a point of the contour; $\mathbf{n}$ is the interior normal; $\mathbf{M}(\mathbf{Z}, \tau)$ is the complex Green's function.

Knowing the function $\mathbf{F}(\mathbf{Z}_3)$, by formula (6) we find the analytic function $\psi(\mathbf{Z}_3)$.

Eliminating after that $\overline{\phi_2(t_2)}$ from the boundary conditions (5), we obtain

$$\phi_2(t_2) = a\phi_1(t_1) + b\overline{\phi_1(t_1)} + c(t_1) \qquad \left(t_1 \in L^1\right), \tag{10}$$

where $$a = \frac{\overline{s_2} - s_1}{s_2 - \overline{s_2}}, \quad b = \frac{\overline{s_2} - \overline{s_1}}{s_2 - \overline{s_2}};$$

$c(t_1)$ is the known function expressed in terms of $f_i(t)$, $g_i(t)$, $\psi(t_3)$ and

$$\sum_{k=1}^{m+1} C_k^{(i)} \beta_k(t) \qquad (i = 1, 2).$$

Replacing in the boundary condition (10) t_1 by t and t_2 by $\alpha(t)$ we have the problem: to find two analytic functions $\phi_1(Z_1)$, $\phi_2(Z_2)$ in the domains D_1, D_2 as well as constants $C_k^{(i)}$ under the boundary condition

$$\phi_2\left[\alpha(t)\right] = a\phi_1(t) + b\overline{\phi_1(t)} + c(t) \quad \left(t \in L^1\right) \tag{11}$$

where $$\alpha(t) = \frac{\left(s_2 - \overline{s_1}\right)t + \left(s_1 - s_2\right)\bar{t}}{s_1 - \overline{s_1}};$$

the function $\alpha(t)$ homeomorphically the contour L_k^1 onto the contour L_k^2 and preserves the direction of the circuit.

2. Consider problem (11) in the general statement.

Let two complex planes $Z_1 = x_1 + iy_1$ and $Z_2 = x_2 + iy_2$ be given and $Z_2 = \alpha(Z_1)$ is the homeomorphism preserving the orientation of the Z_1 plane onto the Z_2 plane. We take a $(m+1)$ – connected domain D_1 in the Z_1 plane bounded by the Ljapunov contour L^1 consisting of closed smooth non-intersecting contours $L_1^1, L_2^1, \ldots, L_m^1, L_{m+1}^1$ of which L_{m+1}^1 contains all the others.

Let the domain D_2 bounded by smooth closed contours $L_1^2, L_2^2, ..., L_m^2, L_{m+1}^2$ not intersecting one another, the last of which encloses all the others and corresponds to the domain D_1 under the transformation $Z_2 = \alpha(Z_1)$.

Assume that a derivative $\alpha'(t)$ is deferent from zero H-continuous.

Let us consider the following boundary value problem: to find two functions $\phi_1(Z_1)$, $\phi_2(Z_2)$ analytic in D_1 and D_2 and H-continuous in $D_j + L^j$ according to the boundary condition

$$\phi_2[\alpha(t)] = a(t)\phi_1(t) + b(t)\overline{\phi_1(t)} + c(t) \quad (t \in L^1), \tag{12}$$

where the functions $a(t)$, $b(t)$, $c(t)$ satisfy the Hölder condition on L^1.

It is possible to prove the following lemma.

Lemma: If $a(t)$ preserves the direction of the circuit on L^1, then the functions $\phi_1(Z_1)$, $\phi_2(Z_2)$ analytic in D_1 and D_2 can be represented in the form

$$\phi_1(Z_1) = -\frac{1}{\pi i}\int_{L^1} \frac{\overline{\phi(\tau)d\tau}}{\tau - Z_1} - \int_{L_m^1} \overline{\phi(\tau)}\left[1 + |\alpha'(\tau)|\right]d\sigma$$

$$(Z_1 \in D_1),$$

$$\phi_2(Z_2) = -\frac{1}{\pi i}\int_{L^1} \frac{\phi[\beta(\tau)]d\tau}{\tau - Z_2} + \int_{L_m^1} \phi(\tau)\left[1 + |\alpha'(\tau)|\right]d\sigma$$

$$(Z_2 \in D_2), \tag{13}$$

where σ is the arc coordinate of the point τ on the contour L^1; $\beta(\tau)$ is the inverse of the function $a(t)$; the density $\phi(\tau)$ is determined to within a constant term of the form $\sum_{k=1}^{m-1} \lambda_k \beta_k(t)$, where λ_k are arbitrary complex constants.

With the aid of the integral representation (13) we reduce the boundary value problem (12) to the singular integral equation

$$1+b(t)\phi(t)+a(t)\overline{\phi(t)}+\frac{1}{\pi i}\int_{L^1}\left[\frac{\alpha'(\tau)}{\alpha(\tau)-\alpha(t)}-\frac{b(t)\overline{\tau'^2(\sigma)}}{\bar{\tau}-\bar{t}}\right]\phi(\tau)d\tau+\frac{a(t)}{\pi i}\int_{L^1}\frac{\overline{\phi(\tau)}d\bar{\tau}}{\bar{\tau}-\bar{t}}+\left[1+b(t)\right]\times$$

$$\times\int_{L_m^1}\phi(\tau)\left[1+|\alpha'(\tau)|\right]d\sigma+a(t)\int_{L_m^1}\overline{\phi(\tau)}[1+|\alpha'(\tau)|$$

$$+|\alpha'(\tau)|\Big]d\sigma=c(t)\qquad\left(t\in L^1\right). \tag{14}$$

The index of this equation over the field of real numbers is equal to 2ind **b(t)** [6]. The boundary condition of the problem adjoint to (12) is written in the form

$$\psi_1(t)=a(t)\alpha'(t)\psi_2\left[\alpha(t)\right]+\overline{b(t)t'^2(s)\alpha'(t)\psi_2\left[\alpha(t)\right]}$$

$$\left(t\in L_1\right) \tag{15}$$

Let l and l' be the numbers of linearly independent solutions of the homogenous problem (12) and the adjoint problm (15) then it be shown that

$$l-l'=2\text{ind }b(t)-2m+2. \tag{16}$$

For solvability of equation (14) it is necessary and sufficient that there hold the conditions

$$\text{Re}\int_{L^1}c(t)\psi_2^{(k)}\left[\alpha(t)\right]\alpha'(t)dt=0\quad\left(k=1,2,...,l'\right), \tag{17}$$

where $\left\{\psi_j^{(k)}(t)\right\}$ is the complex system of linearly independent solutions of the adjoint problem (15).

3. Let us apply the results obtained in 2. for the investigation of the boundary value problem (10) which corresponds to the first fundamental problem of the thermoelasticity. As in this case **b(t)** is a constant, then ind $b(t)=0$ and hence

$$l'=l+2m-2. \tag{18}$$

Let us find l. By virtue of the uniqueness of the solution of the first basic problem of the thermoelasticity the homogenous problem (10) has the solution

$$\phi_j\left(Z_j\right)=i\,A\,h_j\,Z_j+B_j\qquad\left(j=1,\,2\right), \tag{19}$$

where $\mathbf{h}_j$ are known constants [3]; $\mathbf{A}$ is a real constant and $\mathbf{B}_j$ are complex constants. The constant $\mathbf{B}_1$, is determined by the boundary conditions

$$\mathrm{Re}\left\{\phi_1(t)s_1^{j-1}+s_2^{j-1}\phi_2\left[\alpha(t)\right]\right\}-C_j, \tag{20}$$

where $\mathbf{C}_j$ are real constants.

The constant $\mathbf{A}, \mathbf{B}_2=\mathbf{B}_2'+i\mathbf{B}_2''$ remain arbitrary and so the number of linearly independent solutions of the homogenous problem (10) is $\mathbf{l}=3$. Hence the number of linearly independent solutions of the adjoint problem (15) is $\mathbf{l}'=2\mathbf{m}+1$. Let us establish the solvability conditions of the problem (10). From the formulas (17) it follows that the solvability conditions are of the form

$$\mathrm{Re}\int_{L^1} c(t)\psi_2^{(k)}\left[\alpha(t)\right]\alpha'(t)\,dt=0 \qquad (k=1, 2, ..., 2m+1). \tag{21}$$

The functions $c(t)$ is expressed linearly in terms of real constants C_k^j $(k=1,2,...,m+1)$ which must be determined.

The conclusion.

The obtained results permit to show that the system (20) has a solution when the resultant moment of external forces is equal to zero.

References

1. Carslow H.S., Jaeger I.C. Conduction of Heat in Solids, 2nd edn, Clarendon Press, Oxford, 1959. – 484 p.
2. Лехницкий С.Г. Теория упругости анизотропного тела. – Москва: Изд-во «Наука», 1977. – 415 с.
3. Мусхелишвили Н.И. Некоторые основные задачи математической теории упругости. – Москва: Изд-во «Наука», 1966. – 707 с.
4. Подстригач Я.С., Коляно Ю.М. Неустановившиеся температурные поля и напряжения в тонких пластинках. – Киев: Изд-во «Наукова думка», 1972. – 307 с.
5. Уздалев А.И.Некоторые задачи термоупругости для анизотропного тела. – Саратов: Изд-во Саратовского университета, 1967. – 276 с.
6. Литвинчук Г.С., Хасабов Е.Г. Один класс сингулярных интегральных уравнений и граничная задача типа Карлемана. – Казань: Известия высших учебных заведений. Математика, 1964. – 99-110 с.
7. Gavdzinski V.N., El-Sheikh M.G., Maltseva E.V. On the justification of approximate unbounded solutions of mixed plane boundary value problem. Mathematics and computers in simulations. 59, 2002. – 533-539 p.