UDC 531.381

PROBLEMS OF EVOLUTION OF RIGID BODY MOTION SIMILAR TO LAGRANGE TOP

D. Leshchenko[1], T. Kozachenko[1]
[1]*Odesa state academy of civil engineering and architecture*

Abstract. The problem of evolution of the rigid body rotations about a fixed point continues to attract the attention of researches. In many cases, the motion in the Lagrange case can be regarded as a generating motion of the rigid body. In this case the body is assumed to have a fixed point and to be in the gravitational field, with the center of mass of the body and the fixed point both lying on the dynamic symmetry axes of the body. A restoring torque, analogues to the moment of the gravity forces, is created by the aerodynamic forces acting on the body in the gas flow. Therefore, the motions, close to the Lagrange case, have been investigated in a number of works on the aircraft dynamics, where various perturbation torques were taken into account in addition to the restoring torque.

Many works have studied the rotational motion of a heavy rigid body about a fixed point under the action of perturbation and restoring torques. The correction of the studied models is carried out by taking into account external and internal perturbation factors of various physical nature as well as relevant assumptions according to unperturbed motion.

The results of reviewed works may be of interest to specialists in the field of rigid body dynamics, gyroscopy, and applications of asymptotic methods. The authors of this papers present a new approach for the investigation of perturbed motions of Lagrange top for perturbations which assumes averaging with respect to the phase of the nutation angle. Nonlinear equations of motions are simplified and solved explicitly, so that the description of motion is obtained.

Asymptotic approach permits to obtain some qualitative results and to describe evolution of rigid body motion using simplified averaged equations. Thus it is possible to avoid numerical integration. The authors present a unified approach to the dynamics of angular motions of rigid bodies subject to perturbation torques of different physical nature. These papers contains both the basic foundations of the rigid body dynamics and the application of the asymptotic method of averaging. The approach based on the averaging procedure is applicable to rigid bodies closed to Lagrange gyroscope.

The presented brief survey does not purport to be complete and can be expanded. However, it is clear from this survey that there is an literature on the dynamics of rigid body moving about a fixed point under the influence of perturbation torques of various physical nature. The research in this area is in connection with the problems of motion of flying vehicles, gyroscopes, and other objects of modern technology.

Keywords: rigid body, Lagrange's case, rotation, perturbation torque, restoring torque.

ПРОБЛЕМИ ЕВОЛЮЦІЇ РУХУ ТВЕРДОГО ТІЛА, БЛИЗЬКОГО ДО ВОВЧКА ЛАГРАНЖА

Лещенко Д. Д.[1], Козаченко Т. О.[1]
[1]*Одеська державна академія будівництва та архітектури*

Анотація. Проблема еволюції обертань твердого тіла навколо нерухомої точки продовжує привертати увагу дослідників. В багатьох випадках рух в випадку Лагранжа може розглядатися як породжувальний рух твердого тіла. В цьому випадку тіло має нерухому точку і знаходиться в гравітаційному полі з центром мас тіла та нерухомою точкою, які обидва лежать на осі динамічної симетрії тіла. Відновлюючий момент аналогічний гравітаційному моменту створюються аеродинамічними силами, які діють на тіло в потоці газу. Таким чином, рухи твердого тіла, які близькі до випадку Лагранжа, досліджуються в численних роботах з динаміки літальних апаратів, де різні збурення розглядаються разом з відновлюючим моментом сил.

В багатьох роботах досліджується обертальний рух твердого тіла навколо нерухомої точки під дією збурюючих та відновлюваних моментів сил. Уточнення досліджуваних моделей проводиться з розгляданням зовнішніх та внутрішніх збурюючих факторів різної фізичної природи так і різних припущень відносно породжувального руху. Результати приведених робіт можуть бути корисними для спеціалістів з динаміки твердого тіла та застосувань асимптотичних методів.

Автори цих статей представляють новий підхід для дослідження збурених рухів вовчка Лагранжа для збурень, які припускають усереднення по фазі кута нутації. Нелінійні рівняння руху спрощуються та розв’язуються, та описується рух тіла. Асимптотичний підхід дозволяє одержати деякі якісні результати та списати еволюцію руху твердого тіла з допомогою спрощених усереднених рівнянь. Потім здійснюється чисельне інтегрування. Автори представляють уніфікований підхід до динаміки рухів твердого тіла під дією моментів сил різної фізичної природи. Ці статті представляють як базові основи динаміки твердого тіла так і застосування асимптотичного методу усереднення. Підхід, який базується на процедурі усереднення застосовується до твердих тіл, близьких до гіроскопа Лагранжа.

Наданий короткий огляд не претендує на повноту і може бути розширеним. Однак з цього огляду ми бачимо наявність літератури з динаміки твердого тіла, що рухається навколо нерухомої точки під дією збурюючих моментів сил різної фізичної природи. Дослідження в цій галузі знаходять застосування в динаміці літальних апаратів, гіроскопів та інших об’єктів сучасної техніки.

Ключові слова: тверде тіло, випадок Лагранжа, обертання, збурюючий момент, відновлюючий момент.

1 INTRODUCTION

At present in dynamics of a rigid body with fixed point there is bibliography on the theoretical researches of the perturbed motions, that are close to Lagrange case, and on the applications to dynamics of space vehicle and flying machines, of gyrosystems and other engineering objects. Here the brief survey is given, that is devoted to the investigations results for indicated problems. Only the papers are mentioned here, that are the most close to the results of author and his colleagues.

2 LITERATURE DATCAE ANALYSIS AND TARGET SETTINGS

One of the significant areas of investigation in mechanics is a rigid body's motion about a fixed point. As one of the fundamental problems in dynamics it caught interest of well-known scientists throughout the history of its development.

A high volume of works exists regarding the perturbed motions close to Lagrange top, as well as application in the problems of flying vehicles' entry into atmosphere [1, 2], rotating projectile's motion [3], gyroscopy [4-8].

Problems in terms of their theoretical aspect attract attention of specialists in the field of theoretical mechanics. The framework of dynamic unperturbed rigid body models – Lagrange case – allows rigorous formulation of the problems. The refinements of the models under investigation takes place taking into consideration the perturbation torques of different physical nature, both internal and external, and the corresponding suppositions regarding the unperturbed solution.

The mathematical description of symmetrical top motion in the field of gravity is a solved problem in the dynamics of a rigid body. Solution to this problem was first obtained by Lagrange and published in 1788. Many advanced treatises of classical mechanics include this problem [3, 5-10].

3 PURPOSE AND OBJECTIVES OF RESEARCH

We consider the evolution of the dynamics of rigid body motion about a fixed point under the various perturbation torques. The basic method applied in these studies is the Krylov-Bogoliubov asymptotic averaging method.

4 RESEARCH RESULTS

The motions similar to Lagrange case were analyzed in several works in dynamics, in the given works perturbation torques were taken into consideration with restoring torque. The investigations of rigid body dynamics can find application in the area of astronautics.

V.S. Aslanov's monograph [2] studies the rigid body's motion in the atmosphere under the action of biharmonic air dynamic torque and small perturbations. In this paper, he notes the resemblance between the heavy rigid body and the rigid body in a resisting medium (planet atmosphere).

Numerous works [1, 2, 5, 11-22, 27, 29-31, 37, 39, 42-44] have analyzed the perturbed motions of a rigid body similar to Lagrange top.

Works [5, 11] describe the first approximation for the averaging procedure for slow variables of a perturbed motion of a rigid body close to Lagrange's case. In many cases, applied problems permit averaging over the phase of nutation angle. A perturbed motion close to Lagrange's case is analyzed taking into consideration the torques that affect the rigid body from external medium.

Paper [12] investigates the evolution of the rigid body's motion under the effect of an unsteady perturbation torque, the rigid body being close to the Lagrange gyroscope. The concept of the given problem is analyzed in article [13], when restoring and perturbation torques vary slowly in time. The primary objective of this study is to broaden the results of works [5, 11-13] for the problem of dynamically symmetric rigid body motion under the action of restoring and perturbation torques independent or dependent on slow time.

Paper [14] considers the perturbed fast rotation of a rigid body which is close to regular precession in the Lagrange case. Work [15] describes a more general occurrence of the evolution of rotations, where the value of the restoring torque is dependent on the nutation angle.

Paper [16] analyses the perturbed motion of the rigid body, similar to regular Lagrangian precession, affected by slowly time-varying perturbation torque, as well as restoring torque, dependent on the nutation angle. Papers [17, 18] research the evolution of rotations of a rigid body, similar to regular precession, influenced by a restoring torque, dependent on slow time as well as nutation angle; and by a perturbation torque that slowly varies in time.

We presented in [20] some new qualitative and quantitative results of fast motion of a heavy top subject to small perturbation top subject to small perturbation torques. We suggested a new procedure of the averaging method, different from works [5, (sections 4.8.2, 11.3, 11.3.2), 14]. Works [5, 11-20] provide an overview of the received results in rigid body dynamics, as well as a bibliography.

Paper [21] analyzed a symmetric rigid body's motion, similar to the case of Lagrange, influenced by perturbation torques, Newtonian force field and gyro moment vector. It has been endeavored to utilize the averaging procedure in regard to the nutation phase angle, proposed in works [5, 11]. The averaging procedure suggested for investigation of the Lagrange's top fast rotation in works [5, 14] was applied for analysis of rigid body's rotational motion in article [22], in presence of Newtonian field of force, gyro and perturbation torques.

When axisymmetric magnetized body moves in constant field, close to regular precession, the following equations coincide: motion of the satellite to motion of the Lagrange gyroscope. It is known that a dynamically symmetric satellite moves the same way as a heavy rigid body in the Lagrange case, once the satellite possesses a magnetic torque moved along dynamic symmetry axis [23].

The resemblance of the problem of Lagrange's top motion in case of potential perturbations to the problem of satellite's rotation can be observed. The latter's mass center repositions in the equatorial plane's circular orbit, being affected by the Earth's magnetic field [24-26]. Article [27] indicates new results of negligibly asymmetric heavy top's motion, subject to small viscous damping.

While studying a heavy unbalanced gyrostat's motion with an arbitrary torque of internal interaction [28], equations of motion first integrals coincide with corresponding first integrals of rigid body motion in Lagrange's case.

Paper [29] considers heavy symmetric rigid body's motion, the body having a fixed point under effect of frictional forces originated from the surrounding dissipative medium.

In the works [5 (Sections 4.8.2, 11.3), 14-18, 22] the perturbed fast rotational motions of a rigid body, close to regular precession in Lagrange's case, were studied for different orders of smallness of the projections of the perturbation torque vector. In work [5 (Section 4.8.3)], the perturbation torques are small compared to the restoring one. In contrast to work [5 (Sections 4.8.1, 11.1, 11.2), 11-13, 21, studies 14-18, 22] considered the case of a rigid body that rotates rapidly about the axis of dynamic symmetry, and therefore the unperturbed solution was not the trajectory of motion in Lagrange case, but rather some simpler solution.

In paper [30] the author analyzed lower-order resonances throughout Lagrange top's motion, having small mass asymmetry. In article [31] a case similar to Lagrange top was

https://doi.org/10.31650/2618-0650-2022-4-1-23-31

explored, where secondary resonance effects in the spherical motion of a heavy asymmetrical rigid body with moving masses were investigated.

Interest to rigid body rotation about a fixed point attracts a wide circle of specialists, and not only in rigid body dynamics, but also in control theory [32], hydrodynamics [33], physics [34], and elasticity theory [35].

Dissipation is an important factor of determination of heavy symmetric top's motion. In work [36] dragging is estimated with simple models, and is investigated as torque in Euler equations to be solved numerically. In article [37] the authors considered rotation about a fixed point of a heavy dynamically symmetric rigid body with arbitrary asymmetric cavity completely filled with ideal fluid in a resisting medium. The condition of asymptotic stability of the uniform rotations of an asymmetric rigid body in a resisting medium was obtained in [38]. Paper [39] studies heavy symmetrical top's motion, with a cavity filled with viscous fluid, when the axis of the top is diverged from vertical.

In article [40] authors compute in the Lagrange case the Euler angles of precession ψ and proper rotation φ in actual form through hypergeometric functions. The motion of symmetrical rigid body without weight under viscous dissipation was studied. Author of work [41] considers an analytical solution for the dynamics of axially symmetric rotating objects. This work provides the gyroscopic effects theory, elaborating on their physics and utilizing mathematical models of Euler's form for the motion of non-fixed spinning objects.

Paper [42] investigates the question of monoaxial attitude control of a rigid body subject to nonstationary perturbations. The control torque includes a dissipative and a restoring component. The paper analyzes cases of linear and non-linear restoring of perturbation torques. Article [43] explores a top's global asymptotic stabilization to a constant rotation about axes of symmetry. Paper [44] studies heavy Lagrange's top's motion with imbalance of equatorial moments of inertia.

Results and diverse methodologies applied in the rigid body dynamics, as well as investigation of the Lagrange top was studied in works [3, 5-10, 45]. A series of books and papers are dedicated to dynamics of a rigid body in a resistant medium (see, for example, works [5, 7, 8, 11-14, 17, 18, 21, 22]).

It becomes evident from the given analysis that there exists numerous papers on dynamics of a rigid body under the action of perturbation torques of diverse physical nature. The research in the given field is linked to problems of motion of gyroscopes, flying vehicles, as well as other modern technology devices.

5 CONCLUSIONS

For all cases of motion considered in the paper, the authors present and analyze they basic equations of motion. As a result of analysis of solution of the obtained equations, establish some quantitave and qualitative features of the motions and provide a description of the evolution of the body motion. The presentation is illustrated by some examples.

References

1. Kuzmak, G. E. (1970). *The Dynamics of an Uncontrolled Motion of a Vehicle during Atmospheric Re-entry*. Moscow: Nauka. [in Russian].
2. Aslanov, V. S. (2017). *Rigid Body Dynamics for Space Applications*. Oxford: Butterworth Heinemann.
3. Okunev, B. N. (1950). *The Free Motion of a Gyroscope*. Moscow: GITTL. [in Russian].
4. Koshlyakov, V. N. (1985). *Problems in Dynamics of Solid Bodies and in Applied Gyroscope Theory*. Moscow: Nauka. [in Russian].
5. Chernousko, F. L., Akulenko, L. D., Leshchenko, D. D. (2017). *Evolution of Motions of a Rigid Body About its Center of Mass*. Cham: Springer International Publishing.

6. Arnold, R. N., Maunder, L. (1961). *Gyrodynamics and its Engineering Applications*. New York: Academic Press.
7. Magnus, K. (1971). *Kreisel. Theorie and Anwendungen*. New York: Springer-Verlag.
8. Leimanis, E. (1965). *The General Problem of the Motion of Coupled Rigid Bodies about a Fixed Point*. Berlin: Springer.
9. Macmillan, W. D. (1936). *Theoretical Mechanics. Dynamics of Rigid Bodi*es. New York: Mc Graw-Hill.
10. Wittenburg, J. (2008). *Dynamics of Multibody Systems*. Berlin: Springer.
11. Akulenko, L. D., Leshchenko, D. D., Chernousko, F. L. (1979). Perturbed motions of a rigid body, close to the Lagrange case. *Journal of Applied Mathematics and Mechanics*. 43(5). 829–837.
12. Akulenko, L. D., Zinkevich, Ya. S., Kozachenko, T. A., Leshchenko, D. D. (2017). The evolution of motions of a rigid body close to the Lagrange case under the action of an unsteady torque. *Journal of Applied Mathematics and Mechanics*. 82(2). 79–84.
13. Leshchenko, D., Ershkov, S., Kozachenko, T. (2021). Evolution of a heavy rigid body rotation under the action of unsteady restoring and perturbation torques. *Nonlinear Dynamics*. 103. 1517–1528.
14. Akulenko, L. D., Leshchenko, D. D., Chernousko, F. L. (1986). Perturbed motions of a rigid body that are close to regular precession. *Mechanics of Solids.* 21(5). 1–8.
15. Leshchenko, D. D. (1990). Pertubed rotationnal motion of a rigid body. In: Borne P. and Matrosov V. (ed.) The Lyapunov Function Method and Applications. pp. 227-232. Baltzer, J. C. AG, Scientific Publishing, IMACS, Basel.
16. Akulenko, L., Leshchenko, D., Kushpil, T., et al. (2001). Problems of evolution of rotations of a rigid body under the action of perturbing moments. *Multibody System Dynamics.* 6(1). 3–16.
17. Akulenko, L. D., Kozachenko, T. A., Leshchenko, D. D. (2003). Rotations of a rigid body under the action of unsteady restoring and perturbation torques. *Mechanics of Solids.* 38(2). 1–7.
18. Akulenko, L. D., Kozachenko, T. A., Leshchenko, D. D. (2002). Evolution of rotations of a rigid body under the action of restoring and control moments. *Journal of Computer and System Sciences International*. 41(5). 868–874.
19. Ershkov, S. V., Leshchenko, D. D. (2019). On a new type of solving procedure for Euler – Poisson equations (rigid body rotation over a fixed point). *Acta Mechanica.* 230(3). 871–883.
20. Leshchenko, D., Ershkov, S., Kozachenko, T. (2022). Rotations of a rigid body close to the Lagrange case under the action of nonstationary perturbation torque. *Journal of Applied and Computational Mechanics*. 8(3). 1023–1031.
21. Amer, W. S. (2019). The dynamical motion of a gyroscope subjected to applied moments. *Results in Physics.* 12. 1429–1435.
22. Abady, I. M., Amer, T. S. (2018). On the motion of a gyro in the presence of a Newtonian force field and applied moments. *Mathematics and Mechanics of Solids*. 23(9). 1263–1273.
23. Beletsky, V. V., Khentov, A. A. (1985). *Rotational Motion of a Magnetized Satellite*. Moscow: Nauka. [in Russian].
24. Aksenenkova, I. M. (1991). On the influence of geomagnetic field on periodic motion of a satellite relative to its center of mass. *Kosmicheskie Issledovania.* 29(1). 145–148. [in Russian].
25. Konkina, L. I. (1996). Conditionally periodic solutions for the problem of rotation of a magnetized satellite in magnetic field. *Kosmicheskie Issledovania.* 34(4). 442–444. [in Russian].
26. Sidorenko, V. V. (2002). One class of motions for a satellite carrying a strong magnet. *Cosmic Research*. 40(2). 133–141.
27. Sidorenko, V. V. (1994). Capture and escape from resonance in the dynamics of the rigid body in viscous medium. *Journal of Nonlinear Science.* 4(1). 35–57.
28. Aslanov, V. S., Doroshin, A. V. (2006). Two cases of motion of an unbalanced gyrostat. *Mechanics of Solids.* 41(4). 29–39.
29. Simpson, H. C., Gunzburger, M. D. (1986). A two time scale analysis of gyroscopic motion with friction. *Zeitschrift fur angewandte Mathematik und Physik.* 37(6). 867–894.
30. Zabolotnov, Yu. M. (2019). Resonant motions of the statically stable Lagrange spinning top. *Mechanics of Solids.* 54 (5). 652–668.
31. Lyubimov, V. V. (2020). Direct and inverse secondary resonance effects in the spherical motion of an asymmetric rigid body with moving masses. *Acta Mechanica.* 231. 4933–4946.

https://doi.org/10.31650/2618-0650-2022-4-1-23-31

32. Akulenko, L. D. (1994). *Problems and Methods of Optimal Control*. Dordrecht-Boston-London: Kluwer.
33. Dolzhansky, F. V. (2013). *Fundamentals of Geophysical Hydrodynamics*. Heidelberg-New York-Dordrecht-London: Springer.
34. Bogoyavlenskii, O. I. (1985). Integrable Euler equations on Lie algebras arising in problems of mathematical physics. *Math USSR Izvestiya*. 25(2). 207–257. [in Russian].
35. Ilyukhin, A. A. (1979). *Spatial Problems of the Nonlinear Theory of Elastic Rods*. Kiev: Naukova dumka. [in Russian].
36. Tanriverdi, V. (2020).Dissipative motion of a spinning heavy symmetric top. *European Journal of Physics.* 41(5). 055001. DOI:10.1088/1361-6404/ab9930.
37. Kononov, Yu. M., Vasylenko, V. Yu. (2019). On the stability of the rotation under the action of constant moment of a Lagrangian top with perfect fluid in a resisting medium. *Proceedings of IAMM of NASU.* 33. 122–131. [in Ukrainian].
38. Kononov, Yu. M., (2021). Stability of a uniform rotation of an asymmetric rigid body in a resisting medium, *International Applied Mechanics*. 57(4). 432-439.
39. Ivashchenko, B. P. (1976). On the motion of a symmetric gyroscope with a cavity filled with viscous fluid. *Dokl Akad Nauk Ukr SSR*. Ser. A 9. 794–797. [in Russian].
40. Scarpello, G. M., Ritelli, D. (2018). Motions about a fixed point by hypergeometric functions: new non-complex analytical solutions and integration of the herpolhode. *Celestial Mechanics and Dynamical Astronomy.* 130(6). 42. DOI:10.1007/s10569-018-9837-5.
41. Usubamatov, R. (2020). *Theory of Gyroscopic Effects for Rotating Objects. Gyroscopic Effects and Applications*. Singapure: Springer Nature Singapure Pte Ltd,.
42. Alexandrov, A., Tikhonov, A. (2018). Rigid body stabilization under time-varying perturbations with zero mean values. *Cybernetics and Physics*. 7(1). 5–10.
43. Wan, C. J., Tsiotras, P., Coppola, V. T., et al. (1997). Global asymptotic stabilization of a spinning top with torque actuators using stereographic projection. *Dynamics and Control*. 7. 215–233.
44. Holmes, P. J., Marsden, J. E. (1983). Horseshoes and Arnold diffusion for Hamiltonian systems on Lie groups. *Indiana University Mathematics Journal.* 32 (2). 273–309.
45. Routh, E. J. (2005). *Advanced Dynamics of a System of Rigid Bodies*. New York: Dover.

Література

1. Кузмак Г.Е. Динамика неуправляемого движения летательных аппаратов при входе в атмосферу. М.: Наука, 1970. 348 с.
2. Aslanov V.S. Rigid Body Dynamics for Space Applications. Oxford: Butterworth Heinemann, 2017.
3. Окунев Б.Н. Свободное движение гироскопа. М.-Л.: Гостехориздат, 1951. 380 с.
4. Кошляков В. Н. Задачи динамики твердого тела и прикладной теории гироскопов: Аналитические методы. М.: Наука, 1985. 288 с.
5. Chernousko F.L., Akulenko L.D., Leshchenko D.D. Evolution of Motions of a Rigid Body About its Center of Mass. Cham: Springer International Publishing, 2017.
6. Arnold R.N., Maunder L. Gyrodynamics and its Engineering Applications. New York: Academic Press, 1961.
7. Magnus K. Kreisel. Theorie and Anwendungen. New York: Springer-Verlag, 1971.
8. Leimanis E. The General Problem of the Motion of Coupled Rigid Bodies about a Fixed Point. Berlin: Springer, 1965.
9. Macmillan W.D. Theoretical Mechanics. Dynamics of Rigid Bodies. New York: Mc Graw-Hill, 1936.
10. Wittenburg J. Dynamics of Multibody Systems. Berlin: Springer, 2008.
11. Akulenko L.D., Leshchenko D.D., Chernousko F.L. Perturbed motions of a rigid body, close to the Lagrange case. Journal of Applied Mathematics and Mechanics,1979. 43(5). P. 829–837.
12. Akulenko L. D., Zinkevich Ya. S., Kozachenko T. A., Leshchenko, D. D. The evolution of motions of a rigid body close to the Lagrange case under the action of an unsteady torque. Journal of Applied Mathematics and Mechanics, 2017. 82(2). P. 79–84.

13. Leshchenko D., Ershkov S., Kozachenko T. Evolution of a heavy rigid body rotation under the action of unsteady restoring and perturbation torques. Nonlinear Dynamics, 2021. 103. P. 1517–1528.
14. Akulenko L.D., Leshchenko D.D., Chernousko F.L. Perturbed motions of a rigid body that are close to regular precession. Mechanics of Solids, 1986. 21(5). P. 1–8.
15. Leshchenko D.D. Pertubed rotationnal motion of a rigid body. In: Borne P. and Matrosov V. (ed.) The Lyapunov Function Method and Applications, 1990. pp. 227-232. Baltzer, J. C. AG, Scientific Publishing, IMACS, Basel.
16. Akulenko L., Leshchenko D., Kushpil T., et al. Problems of evolution of rotations of a rigid body under the action of perturbing moments. Multibody System Dynamics, 2001. 6(1). P. 3–16.
17. Akulenko L.D., Kozachenko T.A., Leshchenko D.D. Rotations of a rigid body under the action of unsteady restoring and perturbation torques. Mechanics of Solids, 2003. 38(2). P. 1–7.
18. Akulenko L. D., Kozachenko T. A., Leshchenko D. D. Evolution of rotations of a rigid body under the action of restoring and control moments. Journal of Computer and System Sciences International, 2002. 41(5). P. 868–874.
19. Ershkov S.V., Leshchenko D. On a new type of solving procedure for Euler – Poisson equations (rigid body rotation over a fixed point). Acta Mechanica, 2019. 230(3). P. 871–883.
20. Leshchenko D., Ershkov S., Kozachenko T. Rotations of a rigid body close to the Lagrange case under the action of nonstationary perturbation torque. Journal of Applied and Computational Mechanics, 2022. 8(3). P. 1023–1031.
21. Amer W.S. The dynamical motion of a gyroscope subjected to applied moments. Results in Physics, 2019. 12. P. 1429–1435.
22. Abady I.M., Amer T.S. On the motion of a gyro in the presence of a Newtonian force field and applied moments. Mathematics and Mechanics of Solids, 2018. 23(9). P. 1263–1273.
23. Белецкий В.В., Хентов А.А. Вращательное движение намагниченного спутника. М.: Наука, 1985. 287 с.
24. Аксененкова И.М. О влиянии геомагнитного поля на периодическое движение спутника относительно его центра масс. Космические исследования, 1991. 29(1). С. 145–148.
25. Конкина Л.И. Условно-периодические решения в задаче о вращении намагниченного спутника в магнитном поле. Космические исследования, 1996. 34(4). С. 442–444.
26. Sidorenko V.V. One class of motions for a satellite carrying a strong magnet. *Cosmic Research*, 2002. 40(2). P. 133–141.
27. Sidorenko V.V. Capture and escape from resonance in the dynamics of the rigid body in viscous medium. Journal of Nonlinear Science, 1994. 4(1). P. 35–57.
28. Aslanov V.S., Doroshin, A.V. Two cases of motion of an unbalanced gyrostat. Mechanics of Solids, 2006. 41(4). P. 29–39.
29. Simpson H.C., Gunzburger M.D. A two time scale analysis of gyroscopic motion with friction. Zeitschrift fur angewandte Mathematik und Physik, 1986. 37(6). P. 867–894.
30. Zabolotnov Yu.M. Resonant motions of the statically stable Lagrange spinning top. Mechanics of Solids, 2019. 54 (5). P. 652–668.
31. Lyubimov V.V. Direct and inverse secondary resonance effects in the spherical motion of an asymmetric rigid body with moving masses. Acta Mechanica, 2020. 231. P. 4933–4946.
32. Akulenko L.D. Problems and Methods of Optimal Control. Dordrecht-Boston-London: Kluwer, 1994.
33. Dolzhansky F.V. Fundamentals of Geophysical Hydrodynamics. Heidelberg-New York-Dordrecht-London: Springer, 2013.
34. Богоявленский О. И. Интегрируемые уравнения Эйлера на алгебрах Ли, возникающие в задачах математической физики. Изв. АН СССР. Сер. матем. 48(5), 1984. С. 883–938.
35. Илюхин А.А. Пространственные задачи нелинейной теории упругих стержней: монография. – К.: Наукова думка, 1979.
36. Tanriverdi V. Dissipative motion of a spinning heavy symmetric top. European Journal of Physics, 2020. 41(5). 055001. DOI:10.1088/1361-6404/ab9930.
37. Кононов Ю.М., Василенко В. Ю. Про стійкість обертання під дією постійного моменту дзиги Лагранжа з ідеальною рідиною в середовищі, що чинить опір. Праці ІПММ НАН України, 2019. Т. 33. С.122–131.

38. Kononov Yu.M. Stability of a uniform rotation of an asymmetric rigid body in a resisting medium, International Applied Mechanics, 2021. 57(4). P. 432-439.
39. Иващенко Б.П. О движении симметричного волчка с полостью, заполненной вязкой жидкостью. Докл. АН УССР. Сер. А, 1976. №9. С. 794–797.
40. Scarpello G.M., Ritelli D. Motions about a fixed point by hypergeometric functions: new non-complex analytical solutions and integration of the herpolhode. Celestial Mechanics and Dynamical Astronomy, 2018. 130(6). 42. DOI:10.1007/s10569-018-9837-5.
41. Usubamatov R. Theory of Gyroscopic Effects for Rotating Objects. Gyroscopic Effects and Applications. Singapure: Springer Nature Singapure Pte Ltd, 2020.
42. Alexandrov A., Tikhonov A. Rigid body stabilization under time-varying perturbations with zero mean values. Cybernetics and Physics, 2018. 7(1). P. 5–10.
43. Wan C.J., Tsiotras P., Coppola V.T., et al. Global asymptotic stabilization of a spinning top with torque actuators using stereographic projection. Dynamics and Control,1997. 7. P. 215–233.
44. Holmes P.J., Marsden J.E. Horseshoes and Arnold diffusion for Hamiltonian systems on Lie groups. Indiana University Mathematics Journal, 1983. 32(2). P. 273–309.
45. Routh E.J. Advanced Dynamics of a System of Rigid Bodies. New York: Dover, 2005.

Dmytro Leshchenko
Odessa State Academy of Civil Engineering and Architecture
DSc. in Physics and Mathematics, Professor
Didrikhson st. 4, Odessa, Ukraine, 65029
leshchenko_d@ukr.net,
ORCID: 0000-0003-2436-221X
Tetiana Kozachenko
Odessa State Academy of Civil Engineering and Architecture
Didrikhson st. 4, Odessa, Ukraine, 65029
PhD in Physics and Mathematics, Associate Professor
kushpil.t.a@gmail.com
ORCID: 0000-0001-9034-3776

For references:
Leshchenko D., Kozachenko T. (2022). Problems of evolution of rigid body motion similar to Lagrange top. Mechanics and Mathematical Methods. 4 (1). 23–31.

Для посилань:
Лещенко Д. Д., Козаченко Т. О. Проблеми еволюції руху твердого тіла, близького до вовчка Лагранжа. Механіка та математичні методи, 2022. Т. 4. № 1. С. 23–31.