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SOME PROBLEMS OF OPTIMIZATION OF ROD SYSTEMS CONTAINING COMPRESSED ELEMENTS USING ADDITIONAL CONSTRAINTS

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Abstract: The article is devoted to the problem of increasing the stability of rod systems containing longitudinally compressed elements. The influence of the imposition of constraints on the behavior of such systems is investigated in order to determine such places for imposing constraints that provide the maximum stability of the system reinforced by the constraint. To get generality, the consideration includes such rod systems that allow various equilibrium configurations, for example, having internal ideal hinges, as well as an arbitrary distribution of longitudinal compressive forces, including leaving some areas free from compression. For the same purpose, the constraints are considered as generalized, producing a reaction with an arbitrary spatial distribution. The paper formulates some general results related to the influence of the introduction of generalized constraints on the critical forces of a rod system with some generalizations related to the extension of the class of rod systems under consideration. Particular attention is paid to the buckling modes in view of their important role as a basis for describing various configurations of the structure. It has been established that the shape of these modes, in particular, the position of their nodes, is essential for finding the optimal position of the constraint. For the case of constraint in the form of a concentrated hinged support, analytical expressions are obtained that represent the derivatives of the critical forces of the system with respect to the coordinate of the support. The case of a multiple critical force, when this derivative, generally speaking, does not exist, is especially considered. These expressions make it possible to qualitatively characterize the optimal position of the support. The application of some of the obtained results is demonstrated by the example of the problem of finding the optimal position of an intermediate hinged support of a two-span rod supported at the ends by elastic hinged supports. These positions are qualitatively described for various values of the stiffness coefficients of the end supports. It has been established that under certain conditions, the optimal positions of the intermediate support correspond to a special semi-curved mode of buckling, in which one of the spans does not bend, but retains its rectilinear equilibrium shape.

Keywords: rod system, critical force, effect of constraint, optimization, semi-curved buckling mode, qualitative sign.

ДЕЯКІ ЗАДАЧІ ОПТИМІЗАЦІЇ СТРИЖНЕВИХ СИСТЕМ, ЩО МІСТЯТЬ СТИСНУТІ ЕЛЕМЕНТИ, ІЗ ЗАСТОСУВАННЯМ ДОДАТКОВИХ В’ЯЗЕЙ

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Анотація: Статтю присвячено актуальній проблемі підвищення стійкості стрижневих систем, що містять поздовжньо стиснуті елементи. Досліджується вплив накладання в’язей на поведінку таких систем з метою визначення таких місць встановлення в’язей, які забезпечують максимальну критичну силу, яка характеризує стійкість системи, підсиленої в’яззю. Для досягнення загальності опису до розгляду залучені такі стрижневі системи, які допускають різні рівноважні конфігурації, наприклад, такі що мають внутрішні ідеальні шарніри, а також довільний розподіл поздовжніх стискаючих сил, у тому числі такий, що залишає деякі ділянки вільними від стиснення. З тією ж метою в’язі розглядаються як узагальнені, які генерують реакцію з довільним просторовим розподілом. У роботі сформульовані деякі загальні результати, що стосуються впливу введення узагальнених в'язей на критичні сили стрижневої системи, з деякими узагальненнями, пов'язаними з розширенням класу стрижневих систем, що розглядаються. Особливу увагу приділено формам втрати стійкості розглядуваних систем через їх важливу роль як базису для опису різних конфігурацій конструкції. Встановлено, що вид цих форм, зокрема, положення їх вузлів, є суттєвим для відшукання оптимального розміщення в'язі. Для випадку в'язі у вигляді зосередженої шарнірної опори отримано аналітичні вирази, що представляють похідні критичних сил системи по координаті опори. Особливо розглянуто випадок кратної критичної сили, коли ця похідна, взагалі кажучи, не існує. Ці вирази дають можливість якісно характеризувати оптимальне положення опори. Застосування деяких з отриманих результатів продемонстровано на прикладі завдання пошуку оптимального положення проміжної шарнірної опори двопрогонового стрижня, опертого по кінцях на пружні шарнірні опори. Якісно описані такі положення для різних значень коефіцієнтів жорсткості кінцевих опор. Встановлено, що за певних умов оптимальним положенням проміжної опори відповідає особлива напівзігнута форма втрати стійкості, в якій один з прольотів не згинається, а зберігає прямолінійну рівноважну форму.

Ключові слова: стрижнева система, критична сила, вплив в'язі, оптимізація, напівзігнута форма втрати стійкості, якісна ознака..

1 INTRODUCTION

When designing and operating various engineering structures, designers often face the problem of ensuring the stability of their elements operating under conditions of longitudinal compression. This raises various optimization problems associated with providing maximum stability at minimum cost. One of these problems is the search for the most advantageous distribution of constraints available to the designer, which provides the maximum possible value of the critical force of the structure. The proposed article is devoted to solving this problem for a linearly elastic rod system reinforced with one constraint. At the same time, the considered rod systems include systems for which, in the absence of external loads, various configurations are possible, in particular, those having internal perfect hinges, as well as systems in which some sections remain free from longitudinal compression.

2 LITERATURE ANALYSIS AND PROBLEM STATEMENT

Many studies have been devoted to the optimization of elastic structures, in which the variables are the properties and distribution of the material, outlines, and other design parameters [1–3]. Among them, there are relatively few works where the optimum is achieved due to the distribution of singularities and, in particular, the distribution of supports [4–6]. Most of the proposed methods for finding optimal structures use universal schemes developed in mathematics and numerical procedures based on them. At the same time, interesting and important qualitative features of the obtained optimal solutions often remain unnoticed. In a range of works [7–12] devoted to the search for the optimal arrangement of supports for compressed rods, a simple and demonstrative approach was proposed and successfully used, which makes it possible to determine this arrangement and reveal interesting and somewhat unexpected qualitative features of the obtained optimal rods. In this paper, this approach is developed taking into account the inclusion in the consideration of such systems, the study of which leads to equations with degenerate operators. The study of the stability of such systems is connected with the well-known problem in algebra of simultaneous diagonalization of two positive semidefinite matrices [13], however, in this paper, special attention is paid to the spectrum of the corresponding eigenvalues and eigenvectors and its changes in accordance with the objectives of the work.

3 THE PURPOSE AND OBJECTIVES OF THE STUDY

The purpose of the proposed work is to determine such a position of a concentrated elastic or rigid hinge support, in which the main critical force of the rod system reaches its maximum value. To do this, taking into account the expansion of the class of linearly elastic systems under consideration, the features of the spectrum of their critical forces and the buckling modes corresponding to them, as well as their change due to the setting of a constraint, are studied. On this basis, results are derived that make it possible to establish some qualitative features of the desired optimal position. Using these signs, in many cases it is possible to determine these positions practically without calculations and a priori qualitatively describe the corresponding buckling mode and estimate the maximum critical force.

4 RESEARCH RESULTS

4.1. Preliminary results. First, we formulate some general results related to the effect of introducing elastic constraints on the critical forces of the rod system, which are necessary for further conclusions.

1.1. Notations and assumptions

S – elastic rod system, including predetermined elastic and rigid constraints, connecting the points of the system to the ground or fixed bodies.

$S^{(1)}$ – system formed from S by the imposition of one additional constraint.

$\boldsymbol{y} = \boldsymbol{y}(M)$ – displacement (configuration, form) of the system – function of the point M, which determines the position of the point M of the system (in the undeformed state $\boldsymbol{y} \equiv 0$).

$\boldsymbol{q} = \boldsymbol{q}(M)$ – load – a function of the point M, which determines the external force applied to the point M; it is assumed that the forces $\boldsymbol{q}$ applied to the rod of the system are perpendicular to the axis of the rod.

$(\boldsymbol{q}, \boldsymbol{y})$ – work of load $\boldsymbol{q} = \boldsymbol{q}(M)$ on displacement $\boldsymbol{y} = \boldsymbol{y}(M)$. If $(\boldsymbol{q}, \boldsymbol{y}) = 0$, it is said that the load $\boldsymbol{q}$ is orthogonal to the displacement $\boldsymbol{y}$, or that the load $\boldsymbol{q}$ is applied in a generalized node of the configuration (form) $\boldsymbol{y}$.

The functions $\boldsymbol{y}$ and $\boldsymbol{q}$ are considered as elements of the linear spaces Y and Q, respectively, having arbitrarily large but identical finite dimensions. This allows us to assume that $\boldsymbol{q} \equiv 0$, if for any $\boldsymbol{y}$ we have the equality $(\boldsymbol{q}, \boldsymbol{y}) = 0$.

$-C\boldsymbol{y}$ – linear operator that defines the internal forces acting on the points of the system in position $\boldsymbol{y}(M)$ (including the reactions of the elastic and rigid constraints belonging to the system, connecting it to the ground). The "–" sign is assigned to reflect the usual property of elastic structures – to generate reactions that counteract the deformation that caused them. All considered elastic systems are assumed to be conservative. Therefore, the operator C, like all other occurring operators, is assumed to be self-adjoint, i.e. satisfying the condition for any $\boldsymbol{y}$ and $\boldsymbol{v}$

$$(C\boldsymbol{y}, \boldsymbol{v}) = (C\boldsymbol{v}, \boldsymbol{y}) \tag{1}$$

expressing the well-known reciprocity theorem.

It is assumed that in the absence of external forces, the system S can have equilibrium configurations different from $\boldsymbol{y} \equiv 0$, for which $C\boldsymbol{y} = 0 \Rightarrow (C\boldsymbol{y}, \boldsymbol{y}) = 0$, but always $(C\boldsymbol{y}, \boldsymbol{y}) \geq 0$, i.e. operator C is non-negative.

If the elements of system S are subjected to compression by a constant load proportional to parameter P, which does not cause deformation of the system at $\boldsymbol{y}(M) \equiv 0$, then operator C changes to $(C - PN)$, where N is some linear operator, which, like C, we will assume non-negative, i.e. $(N\boldsymbol{y}, \boldsymbol{y}) \geq 0$ with $\boldsymbol{y} \neq 0$. The non-negativity also reflects the usual feature of the behavior of a compressed rod, the rotation of which generates a couple acting in the direction of rotation. Non-strict inequality implies the existence of special configurations for which $(N\boldsymbol{y}, \boldsymbol{y}) = 0$ at $\boldsymbol{y} \neq 0$. In this case, $N\boldsymbol{y} = 0$ is necessary, because for a non-negative operator N the Schwartz inequality $|(N\boldsymbol{y}, \boldsymbol{u})| \leq \sqrt{(N\boldsymbol{y}, \boldsymbol{y})}\sqrt{(N\boldsymbol{y}, \boldsymbol{u})}$ is preserved, which implies for any $\boldsymbol{u}$ $(N\boldsymbol{y}, \boldsymbol{u}) = 0$, if $(N\boldsymbol{y}, \boldsymbol{y}) = 0$. A similar conclusion is also valid for the operator C. For the systems considered in this paper, $N\boldsymbol{y}$ is a system of couples arising as a result of the rotation of compressed elements. Therefore, in these special configurations, all compressed segments must not rotate, i.e. on each of them $\boldsymbol{y} \equiv \mathbf{const}$.. The parameter P will be called the compressive force. Let us introduce the notation

W – $\ker N$ – kernel of operator N, i.e. the set of all $\boldsymbol{y}$ for which $N\boldsymbol{y} = 0$.

V – $\ker C$ – kernel of operator C, i.e. the set of all $\boldsymbol{y}$, for which $C\boldsymbol{y} = 0$.

$U = V \bigcap W$ – thesetofall $\boldsymbol{y}$, for which at the same time $C\boldsymbol{y} = N\boldsymbol{y} = 0$.

W, V and U – subspaces of Y. In what follows, all forms belonging to W are called special.

At this stage of consideration, there is no need for any particular differential or integral representation of the introduced operators. For simplicity, we can assume that we are dealing with matrix representations associated with the choice of some bases in the function spaces $\boldsymbol{y}$ and $\boldsymbol{q}$.

1.2. Critical forces and buckling modes. The equilibrium position of a compressed system subjected to load $\boldsymbol{q}$ is determined by the equation

$$-(C - PN)\boldsymbol{y} + \boldsymbol{q} = 0, \tag{2}$$

expressing the equality to zero of the sum of forces applied to each point of the system. The corresponding homogeneous equation

$$(C - PN)\boldsymbol{y} = 0 \tag{3}$$

determines the equilibrium positions of a system S subjected only to longitudinal compression in the absence of an external load $\boldsymbol{q}$. The existence of such positions different from the trivial one $\boldsymbol{y}(M) \equiv 0$ means that this trivial equilibrium ceases to be stable. It is known [14] that if at least one of the operators C and N is nondegenerate, nontrivial solutions of the homogeneous equation allow us to construct a basis in Y, which can serve as a convenient tool for studying the behavior of the systems under consideration. Assuming the degeneration of both operators C and N, the question of constructing the corresponding basis should be studied in more detail.

Let $\boldsymbol{u}_1,\ldots,\boldsymbol{u}_k$ be a basis in U, $\boldsymbol{u}_{k+1},\ldots,\boldsymbol{u}_d$ be functions that complement $\boldsymbol{u}_1,\ldots,\boldsymbol{u}_k$ to a basis in W, and $\boldsymbol{u}_{d+1}, \boldsymbol{u}_{d+2},\ldots$ be functions that complement $\boldsymbol{u}_1,\ldots,\boldsymbol{u}_d$ to a basis in Y. Substitute $\boldsymbol{y} = \sum x_j \boldsymbol{u}_j$ in (3)

$$\sum_{k+1}^{d} x_j C\boldsymbol{u}_j + \sum_{d+1}^{\dim Y} x_j (C - PN)\boldsymbol{u}_j = 0. \tag{4}$$

Calculating the sum of the works of the forces applied to the system on each of the basis displacements $\boldsymbol{u}_i$, we obtain the system of equations

$$\left.\begin{array}{l}
c_{k+1,k+1}x_{k+1} + \ldots + c_{k+1,d}x_d + c_{k+1,d+1}x_{d+1} + c_{k+1,d+2}x_{d+2} + \ldots = 0 \\
c_{k+2,k+1}x_{k+1} + \ldots + c_{k+2,d}x_d + c_{k+2,d+1}x_{d+1} + c_{k+1,d+2}x_{d+2} + \ldots = 0 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
c_{d,k+1}x_{k+1} + \ldots + c_{dd}x_d + c_{d,d+1}x_{d+1} + c_{d,d+2}x_{d+2} + \ldots = 0 \\
c_{d+1,k+1}x_{k+1} + \ldots + c_{d+1,d}x_d + \left(c_{d+1,d+1} - Pn_{d+1,d+1}\right)x_{d+1} + \left(c_{d+1,d+2} - Pn_{d+1,d+2}\right)x_{d+2} + \ldots = 0 \\
c_{d+2,k+1}x_{k+1} + \ldots + c_{d+2,d}x_d + \left(c_{d+2,d+1} - Pn_{d+2,d+1}\right)x_{d+1} + \left(c_{d+2,d+2} - Pn_{d+2,d+2}\right)x_{d+2} + \ldots = 0 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{array}\right\} \tag{5}$$

where $c_{ij} = \left(C\boldsymbol{u}_j, \boldsymbol{u}_i\right)$ is the generalized stiffness coefficient equal to the work of the total elastic reaction $C\boldsymbol{u}_j$, taken with the opposite sign, caused by the j-th basis displacement $\boldsymbol{u}_j$, on the i-th basis displacement $\boldsymbol{u}_i$; $n_{ij} = (N\boldsymbol{u}_j, \boldsymbol{u}_i)$. Here it is taken into account that if at least one of the indices i, j does not exceed $k = \dim U$, $c_{ij} = (C\boldsymbol{u}_j, \boldsymbol{u}_i) = (C\boldsymbol{u}_i, \boldsymbol{u}_j) = 0$, and if at least one of the indices i, j does not exceed $d = \dim W$, $n_{ij} = (N\boldsymbol{u}_j, \boldsymbol{u}_i) = (N\boldsymbol{u}_i, \boldsymbol{u}_j) = 0$.

According to the definition of the sets U and W for any nontrivial linear combination $\boldsymbol{y} = x_{k+1}\boldsymbol{u}_{k+1} + \ldots x_d \boldsymbol{u}_d$ $C\boldsymbol{y} \neq 0$ and the strict inequality $(C\boldsymbol{y}, \boldsymbol{y}) > 0$ holds for $\boldsymbol{y} \neq 0$, which implies the strict positivity of all principal minors of the matrix of system (5) up to the order $d-k$ inclusive [14]

$$\Delta_j = \begin{vmatrix} c_{k+1,k+1} & c_{k+1,k+2} & \cdots & c_{k+1,j} \\ c_{k+2,k+1} & c_{k+2,k+2} & \cdots & c_{k+2,j} \\ \cdots & \cdots & \cdots & \cdots \\ c_{j,k+1} & c_{j,k+2} & \cdots & c_{jj} \end{vmatrix} > 0, \; j = k+1, \ldots, d. \tag{6}$$

Let us perform the elimination of unknowns $x_{k+1}, \ldots, x_d$ in system (5) using the Gauss method. As a result, the matrices corresponding to (6) take a triangular form, where on the main diagonal in the row with the number h there is a positive value $c_{11}^* = \Delta_{k+1} = c_{k+1,k+1}$, $c_{11}^* = \Delta_{k+1} = c_{k+1,k+1}$, $h = 1, \ldots, d-k$ [14].

$$\left.\begin{array}{r} c_{11}^* x_{k+1} + c_{k+1,k+2} x_{k+2} + \ldots + c_{k+1,d} x_d + c_{k+1,d+1} x_{d+1} + c_{k+1,d+2} x_{d+2} + \ldots = 0 \\ c_{22}^* x_{k+2} + \ldots + c_{k+2,d}^* x_d + c_{k+2,d+1}^* x_{d+1} + c_{k+2,d+2}^* x_{d+2} + \ldots = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ c_{d-k,d-k}^* x_d + c_{d,d+1}^* x_{d+1} + c_{d,d+2}^* x_{d+2} + \ldots = 0 \\ \left(c_{d+1,d+1}^* - P n_{d+1,d+1}\right) x_{d+1} + \left(c_{d+1,d+2}^* - P n_{d+1,d+2}\right) x_{d+2} + \ldots = 0 \\ \left(c_{d+2,d+1}^* - P n_{d+2,d+1}\right) x_{d+1} + \left(c_{d+2,d+2}^* - P n_{d+2,d+2}\right) x_{d+2} + \ldots = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right\} \tag{7}$$

Here, the asterisk denotes the matrix elements obtained as a result of the Gaussian elimination procedure. With the exception of the diagonal of the triangular block, the indices in them have a traditional meaning (row number and column number, taking into account k "zero" rows and columns). Moreover, it follows from equality $c_{ji} = c_{ij}$ that for $i, j > d$ $c_{ji}^* = c_{ij}^*$ [14]. In system (7), the equations, from which $x_{k+1}, \ldots, x_d$ are excluded, represent a standard algebraic eigenvalues problem with symmetric matrices $\|c_{ij}^*\|$ and $\|n_{ij}\|$, $i, j > d$, and the $\|n_{ij}\|$ is positive-definite. It is known [14] that there is a discrete set $P_1 \leq P_2 \leq \ldots$ of non-negative values P, which correspond to $\dim Y - d$ linearly independent non-trivial sets $\{x_{d+1}, x_{d+2}, \ldots\}$, which are solutions to system (7). After they are determined, from the first $d-k$ equations (7) $x_{k+1}, \ldots, x_d$ are uniquely determined. Thus, system (5) allows one to determine $\dim Y - d$ linearly independent configurations $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots$ corresponding to its non-trivial solutions, of the form $\boldsymbol{v} = x_{k+1}\boldsymbol{u}_{k+1} + \ldots x_d \boldsymbol{u}_d + x_{d+1}\boldsymbol{u}_{d+1} + \ldots$. It is convenient to take them as basis ones in Y instead of $\boldsymbol{u}_{d+1}, \boldsymbol{u}_{d+2}, \ldots$. They satisfy the equation

$$(C - P_j N)\boldsymbol{v}_j = 0, \tag{8}$$

and also the orthogonality relation $(N\boldsymbol{v}_i,\boldsymbol{v}_j)=0$, if $P_i \neq P_j$, and can be chosen normalized according to the condition $(N\boldsymbol{v}_i,\boldsymbol{v}_i)=1$. In difference to the case of non-degeneracyof N, they cannot be a basis of Y, since their number is d less than the dimensionof Y. Another difference from the standard problem is that the functions $\boldsymbol{v}_j$ are not defined uniquely, because substitution in (3) shows that if $\boldsymbol{v}_j$ is its solution corresponding to $P=P_j$, then $\boldsymbol{v}_j^*=g_1\boldsymbol{u}_1+\ldots g_k\boldsymbol{u}_k+\boldsymbol{v}_j$ for any $g_1,\ldots,g_k$ will also be its solution corresponding to the same $P=P_j$. The quantities P_j are called critical forces (hereinafter – CRF), and $\boldsymbol{v}_j$ – the corresponding buckling modes (hereinafter – BM) of system S.

1.3. Expansionof forms by buckling modes. Expansion of an arbitrary system configuration by its BM is an effective tool for solving various problems of stability theory. For the degenerate N, one can construct a similar generalized expansion by supplementing the set of BM $v_1, v_2, \ldots$ with functions $\boldsymbol{u}_1,\ldots,\boldsymbol{u}_d$. In this case, it is convenient to replace the set $\boldsymbol{u}_{k+1},\ldots,\boldsymbol{u}_d$ with their linearly independent combinations $\boldsymbol{w}_1,\ldots,\boldsymbol{w}_{d-k}$ for which $(C\boldsymbol{w}_i,\boldsymbol{w}_j)=0$, and $(C\boldsymbol{w}_j,\boldsymbol{w}_j)=c_j>0$. Andbesides $(C\boldsymbol{v}_i,\boldsymbol{w}_j)=P_j(N\boldsymbol{v}_i,\boldsymbol{w}_j)=P_j(N\boldsymbol{w}_j,\boldsymbol{v}_i)=0$. Anyconfiguration of system S canberepresentedas

$$\boldsymbol{y}=\sum_1^k g_j\boldsymbol{u}_j+\sum_1^{d-k} b_j\boldsymbol{w}_j+\sum a_j\boldsymbol{v}_j\,, \tag{9}$$

where a_j, b_j and g_j are scalars, and

$$a_j=(N\boldsymbol{y},\boldsymbol{v}_j),\ b_j=\frac{1}{c_j}(C\boldsymbol{y},\boldsymbol{w}_j).$$

We use expansion (9) to solve the inhomogeneous problem (2) (longitudinal-transverse bending). Substituting (9) into (2), taking into account $C\boldsymbol{u}_j=0$, $N\boldsymbol{u}_j=N\boldsymbol{w}_j=0$, we obtain

$$-\sum_1^{d-k} b_jC\boldsymbol{w}_j-\sum a_j(C\boldsymbol{v}_j-PN\boldsymbol{v}_j)+\boldsymbol{q}=0 \Rightarrow -\sum_1^{d-k} b_jC\boldsymbol{w}_j-\sum a_j(P_j-P)N\boldsymbol{v}_j+\boldsymbol{q}=0\,. \tag{10}$$

Considering the left side of (10) as the total load applied to the points of the system, and calculating the work of this load on displacements $\boldsymbol{v}_j$, $\boldsymbol{w}_j$ and $\boldsymbol{u}_j$, we find

$$a_j=\frac{(\boldsymbol{q},\boldsymbol{v}_j)}{P_j-P},\ b_j=\frac{(\boldsymbol{q},\boldsymbol{w}_j)}{c_j}. \tag{11}$$

In addition, for the $\boldsymbol{u}_1,\ldots,\boldsymbol{u}_k$ the relations $(\boldsymbol{q},\boldsymbol{u}_j)=0$ are satisfied as necessary conditions for the existence of a solution to Eq. (2). These conditions are also sufficient, since together with conditions (11) they mean the equality to zero of the generalized forces corresponding to all generalized coordinates of the system S. This fact is an expression of the elementary result that a matrix equation $A\boldsymbol{y}=\boldsymbol{q}$ has a solution if and only if its right-hand side is orthogonal in the Euclidean sense to any solution of the equation $A^*\boldsymbol{y}=0$, where A^* is the matrix transposed with respect to A.

If P coincides with one of the CRF P_j of the system S, its equilibrium, as can be seen from (11), is possible only if the load $\boldsymbol{q}$ is orthogonal, $(\boldsymbol{q},\boldsymbol{v}_j)=0$, to all BMs corresponding to P_j. In this case, the system can have infinitely many equilibrium configurations, because, as Eq. (2) shows, along with $\boldsymbol{y}$, the superposition of $\boldsymbol{y}$ and any linear combination of all BMs corresponding to P_j satisfy (2).

1.4. Generalized constraint and generalized flexibility. We will say that one constraint is imposed on the system S if at some points of the system another elastic system is attached to it, which, for any joint displacements $\boldsymbol{y}=\boldsymbol{y}(M)$, acts on the system S with a load $\boldsymbol{R}=\boldsymbol{R}(M)$ proportional to some function $\boldsymbol{r}=\boldsymbol{r}(M)$ taken as a unit (basis); thus, in any position of the system $\boldsymbol{R}=-R\boldsymbol{r}$, where $\boldsymbol{r}=\boldsymbol{r}(M)$ does not depend on this position and is a characteristic of a particular constraint. In the case of a point support, the load $\boldsymbol{r}=\boldsymbol{r}(M)$ is one concentrated force applied at that point and equal in magnitude to the accepted unit of force.

Along with the spatial distribution $\boldsymbol{r}=\boldsymbol{r}(M)$ of the basis load, the constraint is characterized by the value of flexibility, which is determined from the following considerations.

Consider the constraint as a separate elastic structure with its own stiffness operator C' and load it with the force $\boldsymbol{R}=C'\boldsymbol{u}$, where $\boldsymbol{u}=\boldsymbol{u}(M)$ is the corresponding configuration of the constraint, which is not defined uniquely. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are different configurations corresponding to the same loading $\boldsymbol{R}=C'\boldsymbol{u}=C'\boldsymbol{v}$, then due to self-adjointness (1) $(\boldsymbol{R},\boldsymbol{u})=(C'\boldsymbol{u},\boldsymbol{u})=(C'\boldsymbol{v},\boldsymbol{u})=(C'\boldsymbol{u},\boldsymbol{v})=(\boldsymbol{R},\boldsymbol{v})$, i.e. the work of the load $\boldsymbol{R}$ on all the displacements it causes is the same. Since $(C'\boldsymbol{u},\boldsymbol{u})$ represents twice the potential energy of the constraint at the position $\boldsymbol{u}$, this means that in all positions of the constraint caused by its loading $\boldsymbol{r}$ (and generating a reactive load $-\boldsymbol{r}$), it has the same potential energy.Then in an arbitrary position $\boldsymbol{u}$ $(\boldsymbol{R},\boldsymbol{u})=R(\boldsymbol{r},\boldsymbol{u})=R^2(\boldsymbol{r},R^{-1}\boldsymbol{u})=\delta R^2$, where δ does not depend on this position and is equal to twice the potential energy of the constraint developing the basis reaction $-\boldsymbol{r}$. In this case, as we see, the numerical value of the reaction is equal to $R=(\boldsymbol{r},\boldsymbol{u})/\delta$. In the case of a point elastic support, δ is equal to the work of a unit force on the displacement of the support caused by it, i.e. this displacement itself, which is called the flexibility of the support. Therefore, for a generalized constraint, we will call the value δ generalized flexibility and consider it as a characteristic of the stiffness of the constraint.

1.5. Influence of constraint on critical forces. The configuration $\boldsymbol{y}$ of the system $S^{(1)}$ formed from S by the imposition of one constraint, at buckling, can be defined as the result of the action of a reactive load $\boldsymbol{R}=-R\boldsymbol{r}$, considered as external one, on the system S released from the constraint. According to (2)

$$-(C-PN)\boldsymbol{y}+\boldsymbol{R}=0 \Rightarrow (C-PN)\boldsymbol{y}+R\boldsymbol{r}=0,\ R=(\boldsymbol{r},\boldsymbol{y})/\delta. \tag{12}$$

The solution of this equation is sought in the form of a generalized expansion (9) by the eigenforms of thesystem S, whose substitution into (12) gives

$$a_j=-\frac{R(\boldsymbol{r},\boldsymbol{v}_j)}{P_j-P},\ b_j=-\frac{R(\boldsymbol{r},\boldsymbol{w}_j)}{c_j}. \tag{13}$$

The presence of forms $\boldsymbol{u}_1,\ldots,\boldsymbol{u}_k$ in the expansion leads to the need to fulfill the relations $R(\boldsymbol{r},\boldsymbol{u}_j)=0$.

If for at least one j $(\boldsymbol{r},\boldsymbol{u}_j)\neq 0$, there must be $R=(\boldsymbol{r},\boldsymbol{y})/\delta=0$ whence, on the basis of (12), follows $(C-PN)\boldsymbol{y}=0$, i.e. $\boldsymbol{y}$ coincides with one of the BMs of system S, and $P=P_j$ is the corresponding CRF.In this case, in expansion (9), the coefficients g_j must satisfy the condition $\sum_1^k g_j(\boldsymbol{r},\boldsymbol{u}_j)=0$. This implies that the spectra of $S^{(1)}$ and S coincide and the

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multiplicity of P_j in the spectrum of $S^{(1)}$ is one less than its multiplicity in S (due to the reduction in the dimension of the set of special forms $\boldsymbol{u}_j$).

Thus, the appearance in the spectrum of the system $S^{(1)}$ of a CRF P that does not coincide with any of the CRFs of the system S is possible only at $R=(\boldsymbol{r},\boldsymbol{y})/\delta \neq 0$, which requires that for all j, $j=1,\ldots,k$, $(\boldsymbol{r},\boldsymbol{u}_j)=0$.In this case, each of the functions $\boldsymbol{u}_1,\ldots,\boldsymbol{u}_k$ satisfies equation (12) for any P and in expansion (9) the coefficients g_j can be arbitrary, and P is not less than $(k+1)$-multiple CRF of the system $S^{(1)}$, which, together with $\boldsymbol{y}$, also corresponds to the forms $\boldsymbol{y}+\sum_1^k g_j\boldsymbol{u}_j$ for arbitrary g_j. If, in addition to the conditions $(\boldsymbol{r},\boldsymbol{u}_j)=0$, the equality $(\boldsymbol{r},\boldsymbol{y})=0$ is satisfied, P coincides with one of the CRFs of S, and $\boldsymbol{y}$ is one of the forms of the system S corresponding to it.

The value P is determined by the following equation of critical forces, which is obtained from the equality $R=(\boldsymbol{r},\boldsymbol{y})/\delta$ if expansion (9) is substituted into it, taking into account relations (13), orthogonality $(C\boldsymbol{v}_i,\boldsymbol{w}_j)=(N\boldsymbol{w}_j,\boldsymbol{v}_i)=0$, and the accepted normalization of $\boldsymbol{v}_j$

$$\left(-\boldsymbol{r},R^{-1}\boldsymbol{y}\right)+\delta=0 \quad \Rightarrow \quad \sum\frac{(\boldsymbol{r},\boldsymbol{v}_j)^2}{P_j-P}+\sum_1^{d-k}\frac{(\boldsymbol{r},\boldsymbol{w}_j)^2}{c_j}+\delta=0, \tag{14}$$

where $\left(-\boldsymbol{r},R^{-1}\boldsymbol{y}\right)$ is the work of the basis reaction $-\boldsymbol{r}$ on the displacement (9) of the system S caused by it.

This equality defines the CRFs of $S^{(1)}$ that were not in the spectrum of S. As we see, for the existence of such CRFs, it is necessary that at least for one of the BMs $\boldsymbol{v}_j$ $(\boldsymbol{r},\boldsymbol{v}_j)$be different from zero. We repeat that (14) is valid only at $(\boldsymbol{r},\boldsymbol{u}_j)=0$ for all j, $j=1,\ldots,k$. If $U=V\bigcap W=0$, this requirement is omitted.

If P, determined from (14), does not coincide with any of the CRFs P_j of the system S, then exactly one non-special form $\boldsymbol{y}$, determined from (9) and (13), corresponds to it up to a term of the form $\sum_1^k g_j\boldsymbol{u}_j$. Otherwise, from two linearly independent non-special BMs satisfying equation (12), one could compose a linear combination satisfying homogeneous equation (3). Therefore, if the multiplicity of P in the spectrum of $S^{(1)}$ is greater than $(k+1)$, P must be one of the CRFs P_i of the system S, which corresponds to a non-special BM, different from $\sum_1^k g_j\boldsymbol{u}_j$. In this case, the equality $(\boldsymbol{r},\boldsymbol{v}_i)=0$ must hold, because only in this case P_i can be the root of equation (14).

If the positions of the points in the undeformed configuration of the system under consideration are determined by the coordinate x, we can assume that $\boldsymbol{y}=\boldsymbol{y}(M)=\boldsymbol{y}(x)$. We assume that the displacements of the points of all rod elements are perpendicular to the undeformed rectilinear axis of each element, parallel to each other, as well as to the forces of all considered loadings $\boldsymbol{q}$, $\boldsymbol{r}$. If $\boldsymbol{r}$ represents a concentrated force equal to one, applied at a point with coordinate s, then the work $(\boldsymbol{r},\boldsymbol{y})$ is numerically equal to the displacement $y(s)$ of this point, provided that the direction of this unit force coincides with the accepted direction of positive displacements. In this case, (14) can be rewritten as

$$\Gamma(s,s,P)+\delta=0 \quad\Rightarrow\quad \sum\frac{v_j^2(s)}{P_j-P}+\sum_1^{d-k}\frac{w_j^2(s)}{c_j}+\delta=0, \tag{15}$$

where $\Gamma(s,s,P)$ is the deflection at a point s of the system S, compressed by the force P, caused by a unit concentrated transverse force applied at that point.

Moreover, the equalities $(\boldsymbol{r},\boldsymbol{u}_j)=0$ mean that $u_j(s)=0$ for all j, i.e., s is the generalized node of all special forms $\boldsymbol{u}_1,\dots,\boldsymbol{u}_k$. This will be the case, for example, if the system S contains a continuous rod, longitudinally compressed along the entire length or part of it and supported in one or more of its cross sections on an elastic or rigid point support. In this case, in any of the forms $\boldsymbol{u}_1,\dots,\boldsymbol{u}_k$, if they exist, $u_j(x)\equiv 0$ on this rod. If $U=V\bigcap W=0$, no restrictions are imposed on the position s of the support in (15).

Equation (15) allows us to get a number of general conclusions regarding the effect of the introduction of constraint on the spectrum of CRS. To this purpose, we represent the solution of equation (15) graphically.

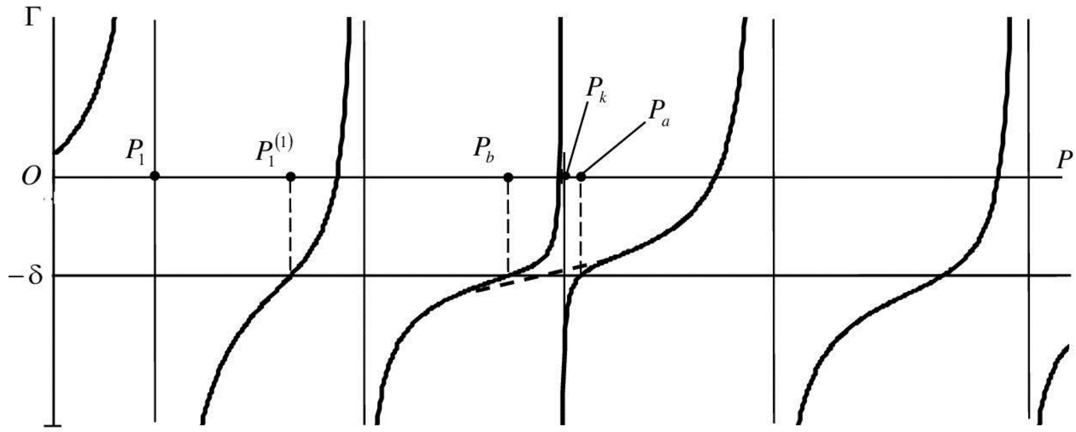

Fig. 1. Graphical representation of the solution of equation (15). $\Gamma=\Gamma(s,s,P)$

Let us focus only on the CRFs, which correspond to non-special BMs $\boldsymbol{v}_j$. We divide the spectrum of the system S into two parts. In one we will include changeable CRFs (CCRFs) P_j, each of which corresponds to at least one BM, not orthogonal to the constraint, for which $(\boldsymbol{r},\boldsymbol{v}_j)\neq 0$. The second includes unchangeable CRFs (UCRFs), which, together with the corresponding BMs, do not change after the introduction of constraint and are present in full in the spectrum of the system $S^{(1)}$. For them $(\boldsymbol{r},\boldsymbol{v}_j)=0$.The condition $(\boldsymbol{r},\boldsymbol{v}_j)\neq 0$ means that CCRFs are poles of $\Gamma(s,s,P)$ as a function of P and on the graph (Fig. 1) they correspond to infinite discontinuities. If the CCRFs in the spectrum of the system S had multiplicity r, then in the spectrum of $S^{(1)}$ its multiplicity is equal to $r-1$, since r linearly independent BMs $\boldsymbol{v}_j$ can be combined into $r-1$ linearly independent combinations orthogonal to $\boldsymbol{r}$.

The spectrum of the system $S^{(1)}$ contains all multiple CCRFs of the system S with a multiplicity one less than their multiplicity in S, all UCRFs of the system S with the same multiplicity as in S and, finally, all the roots of equation (15) (among which there cannot be CCRFs, but there may be UCRFs). Thus, the spectrum of the system $S^{(1)}$ is formed from the spectrum of S by decreasing the multiplicity of each CCRF by one and joining all the roots of equation (15). The number of CRFs of the system $S^{(1)}$ (calculated by their multiplicity) falling on a certain segment of the numerical axis is equal to the number of CRFs of the system S on this segment minus the number of poles and plus the number of roots of the function $\Gamma(s,s,P)+\delta$. Let us denote $n(<P)$ the number of CRFs in the spectrum of the system S strictly less than P; $n^{(1)}(<P)$ is the number of CRFs in the spectrum of the system $S^{(1)}$ strictly less than P.

The entire numerical axis P can be divided into three subsets. The first contains segments from zero to the first pole of $\Gamma(s,s,P)$ and from the roots of equation (15) to the nearest pole on the right, but does not include the ends of these segments. On them $\Gamma(s,s,P)+\delta>0$. The second consists of poles of $\Gamma(s,s,P)$. The third contains segments from the poles to the nearest right root of equation (15) and includes these roots. On them $\Gamma(s,s,P)+\delta\le 0$. It can be seen from the graph that for any P of the first and second subsets, the equality $n^{(1)}(<P)=n(<P)$ holds. On the third subset $n^{(1)}(<P)=n(<P)-1$. The first subset can contain only UCRFs belonging to the spectra of systems S and $S^{(1)}$ with the same multiplicity, for which the equalities $P_j=P_j^{(1)} \Rightarrow P_j=P_j^{(1)}\le P_{j+1}$ hold. The second contains all the CCRFs of the system S, taking into account their multiplicity, and the CRFs of the system $S^{(1)}$ equal to them with a multiplicity one less.They satisfy the relations $P_j=P_j^{(1)}=P_{j+1}$, if $P_j=P_{j+1}$ is a multiple CCRF, and $\Rightarrow P_j\le P_j^{(1)}<P_{j+1}$, if P_{j+1} is a simple CCRF.The third one contains the CRFs belonging to both systems with the same multiplicity, but with a changed numbering, so $P_j^{(1)}=P_{j+1}\Rightarrow P_j\le P_j^{(1)}=P_{j+1}$. In addition to them, the third subset includes all the roots of equation (15). These roots $P_j^{(1)}$ satisfy the relation $P_j<P_j^{(1)}\le P_{j+1}$.

Thus, in all cases CRFs of system $S^{(1)}$ satisfy the well-known estimates [15]

$$P_j\le P_j^{(1)}\le P_{j+1}, \tag{16}$$

that establish the boundaries of their change due to the imposition of the constraint. From them, in particular, it follows that the CRF of the system $S^{(1)}$ cannot exceed the next by number CRF of the system S. When studying the conditions for the maximum increase of the CRF, the following statements, which follow from the previous considerations, are useful.

A. If at least one of the BMs of the system S corresponding to CRF P_{j+1} is not orthogonal to the constraint, strict inequality $P_j^{(1)}<P_{j+1}$ is satisfied.

B. For the maximum increase of the j-th CRF, $P_j^{(1)}=P_{j+1}$, it is necessary that the constraint be orthogonal to each BM corresponding to the $(j+1)$-th CRF of the system S.

The above arguments and conclusions are of a general nature and remain valid if we substitute $\left(-\boldsymbol{r},R^{-1}\boldsymbol{y}\right)$ instead of $\Gamma(s,s,P)$ in them and consider equation (14) instead of (15).

1.6. Changing of the CRF when moving the constraint. Relation (15) makes it possible to trace the change of the critical force P of the system $S^{(1)}$ when the position s of the introduced support changes. Let us differentiate (15) with respect to s

$$2\sum\frac{v_j(s)v_j'(s)}{P_j-P}+\left[\sum\frac{v_j^2(s)}{\left(P_j-P\right)^2}\right]\frac{\partial P}{\partial s}+2\sum_1^{d-k}\frac{w_j(s)w_j'(s)}{c_j}=0\,. \tag{17}$$

At buckling of the system $S^{(1)}$ in form $\boldsymbol{y}$, its point, which has the coordinate x, according to (9) receives displacement

$$y(x)=\sum_{1}^{k} g_j u_j(x)+\sum_{1}^{d-k} b_j w_j(x)+\sum a_j v_j(x).$$

In those areas where $u_j(x)\equiv 0$, this displacement and the slope $y'(x)$ of the section x, taking into account (13), are equal, respectively

$$y(x)=\sum_{1}^{d-k} b_j w_j(x)+\sum a_j v_j(x)=-R\left[\sum_{1}^{d-k}\frac{w_j(s)}{c_j}w_j(x)+\sum\frac{v_j(s)}{P_j-P}v_j(x)\right],$$

$$y'(x)=-R\left[\sum_{1}^{d-k}\frac{w_j(s)}{c_j}w_j'(x)+\sum\frac{v_j(s)}{P_j-P}v_j'(x)\right],$$

and on a support in s

$$y(s)=-R\left[\sum\frac{v_j^2(s)}{P_j-P}+\sum_{1}^{d-k}\frac{w_j^2(s)}{c_j}\right],\; y'(s)=-R\left[\sum_{1}^{d-k}\frac{w_j(s)}{c_j}w_j'(s)+\sum\frac{v_j(s)}{P_j-P}v_j'(s)\right]. \tag{18}$$

As well

$$(N\boldsymbol{y},\boldsymbol{y})=\left(N\sum a_j\boldsymbol{v}_j,\sum a_j\boldsymbol{v}_j\right)=\sum a_j^2=\sum\frac{R^2(\boldsymbol{r},\boldsymbol{v}_j)^2}{(P_j-P)^2}=R^2\sum\frac{v_j^2(s)}{(P_j-P)^2}. \tag{19}$$

This allows us to rewrite (17) as

$$\frac{(N\boldsymbol{y},\boldsymbol{y})}{R^2}\frac{\partial P}{\partial s}-2\frac{y'(s)}{R}=0,$$

whence

$$\frac{\partial P}{\partial s}=2\frac{Ry'(s)}{(N\boldsymbol{y},\boldsymbol{y})}. \tag{20}$$

The form $\boldsymbol{y}$ is determined up to a constant factor, which can be chosen so that the equality $(N\boldsymbol{y},\boldsymbol{y})=2$ holds. Then (20) takes the form

$$\frac{\partial P}{\partial s}=Ry'(s)\Leftrightarrow\frac{\partial P}{\partial s}=cy(s)y'(s), \tag{21}$$

where $c=1/\delta$ is the stiffness coefficient of the introduced support.

Result (21) was known and used earlier for a more bounded class of rod systems [7–12].

Generally speaking, it is not valid if the critical force P is a multiple, since in this case the corresponding BM $y(x)$ and its derivative $y'(x)$ are not uniquely defined.

Relation (21) represents the derivative of that CRF, which is the root of equation (15). As noted, the system $S^{(1)}$ can also have CRFs equal to some critical forces of the system S, provided that the movable support falls in the node s_0 of the corresponding BM of rod S, i.e. P_k at $v_k(s_0)=0$ (don't confuse k and $\dim U$). For them, relation (21) is also valid if P_k is not a root of equation (15). In this case, relation (17) is not valid, because when the coordinate s changes, not only the root changes, but also the form of the equation of CRFs (see (15))

$$\sum_{j\neq k}\frac{v_j^2(s)}{P_j-P}+\frac{v_k^2(s)}{P_k-P}+\sum_{1}^{d-k}\frac{w_j^2(s)}{c_j}+\delta=0. \tag{22}$$

It follows from it

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$$\frac{v_k^2(s)}{P-P_k}=\left[\frac{v_k^2(s)-v_k^2(s_0)}{s-s_0}\right]\bigg/\left[\frac{P-P_k}{s-s_0}\right]=\sum_{j\neq k}\frac{v_j^2(s)}{P_j-P}+\sum_{1}^{d-k}\frac{w_j^2(s)}{c_j}+\delta .$$

Passing to the limit at $s\to s_0$, we get the equality

$$\frac{2v_k(s_0)v_k'(s_0)}{P'(s_0)}=\sum_{j\neq k}\frac{v_j^2(s_0)}{P_j-P_k}+\sum_{1}^{d-k}\frac{w_j^2(s_0)}{c_j}+\delta=\Gamma(s,s,P)+\delta\neq 0,$$

where the prime denotes the derivative with respect to s. It follows from it that $P'(s_0)=0$, in accordance with equality (21), which thus shows that the optimal positions of the movable support should be sought among those points of the system at which the displacement $y(x)$ or slope $y'(x)$ vanishes.

The multiplicity of the CRF in the system $S^{(1)}$ arises, in particular, when equation (15) has a root equal to one of the CRFs P_k of the system S. As stated above, in this case the support must be in the node s_0 of BM $\boldsymbol{v}_k$. Subtracting (15) from (22) and dividing by (P_k-P), we get

$$\frac{v_k^2(s)}{(P_k-P)^2}=-\frac{1}{P_k-P}\left[\sum_{j\neq k}\frac{v_j^2(s)}{P_j-P}-\sum_{j\neq k}\frac{v_j^2(s_0)}{P_j-P_k}\right]-\frac{1}{P_k-P}\left[\sum_{1}^{d-k}\frac{w_j^2(s)-w_j^2(s_0)}{c_j}\right]=$$

$$=-\frac{1}{P_k-P}\left[\sum_{j\neq k}\frac{v_j^2(s_0)}{P_j-P}-\frac{v_j^2(s_0)}{P_j-P_k}\right]-\frac{1}{P_k-P}\left[\sum_{j\neq k}\frac{v_j^2(s)-v_j^2(s_0)}{P_j-P}\right]-$$

$$-\frac{1}{P_k-P}\left[\sum_{1}^{d-k}\frac{w_j^2(s)-w_j^2(s_0)}{c_j}\right]\Rightarrow$$

$$\frac{v_k^2(s)}{(P_k-P)^2}=\sum_{j\neq k}\frac{v_j^2(s_0)}{(P_j-P)(P_j-P_k)}-\frac{1}{P_k-P}\left\{\left[\sum_{j\neq k}\frac{v_j^2(s)-v_j^2(s_0)}{P_j-P}\right]+\left[\sum_{1}^{d-k}\frac{w_j^2(s)-w_j^2(s_0)}{c_j}\right]\right\}. \quad (23)$$

When the support is moved from s_0 to s, instead of a multiple CRF P_k, two different CRFs $P_k^{(1)}=P_b$ and $P_{k+1}^{(1)}=P_a$, $P_a>P_k>P_b$, appear (see Fig. 2), which correspond to BM $\boldsymbol{y}_k$ and $\boldsymbol{y}_{k+1}$, satisfying the orthogonality condition taking into account (13)

$$(N\boldsymbol{y}_k,\boldsymbol{y}_{k+1})=\sum\frac{v_j^2(s)}{(P_j-P_b)(P_j-P_a)}=\sum_{j\neq k}\frac{v_j^2(s)}{(P_j-P_b)(P_j-P_a)}+\frac{v_k^2(s)}{(P_k-P_b)(P_k-P_a)}=0,$$

whence

$$\frac{v_k^2(s)}{(P_k-P_b)(P_k-P_a)}=-\sum_{j\neq k}\frac{v_j^2(s)}{(P_j-P_b)(P_j-P_a)}. \quad (24)$$

When $s\to s_0$, P_b and P_a tend to P_k, $v_k^2(s)/(s-s_0)^2\to v_k'^2(s_0)$, the first term on the right side of (23), taking into account (24), has a limit equal to $-v_k'^2(s_0)/P_b'P_a'$, where P_b' and P_a' are the one-sided derivatives with respect to s of P_b and P_a, respectively, equal to

$$P_a' = \lim_{s \to s_0} \frac{P_a - P_k}{s - s_0}, \; P_b' = \lim_{s \to s_0} \frac{P_b - P_k}{s - s_0}.$$

From (24), taking into account (19), the relation follows

$$\frac{v_k'^2(s_0)}{P_b'P_a'} = -\sum_{j \neq k} \frac{v_j^2(s_0)}{(P_j - P_k)^2} = -\frac{(N\boldsymbol{y}, \boldsymbol{y})}{R^2}, \tag{25}$$

where $\boldsymbol{y}$ is that of the BMs of the system $S^{(1)}$ at the location of the support at the node of the BM $\boldsymbol{v}_k$ corresponding to the CRS P_k, in the expansion of which (9) due to (13) there is no term $a_k\boldsymbol{v}_k$. It means that $(N\boldsymbol{y}, \boldsymbol{v}_k) = 0$.

The sum in (25) is equal to the derivative with respect to P of the deflection of the system $S^{(1)}$ on the support placed at the node of the BM $\boldsymbol{v}_k$, at $P = P_k$ and $R = 1$. It is equal to the tangent of the slope of the graph in Fig. 1 at $P = P_k$ and without discontinuity (see dashed line), i.e. at $s = s_0$.

Substituting $P = P_a$ into (23) and passing to the limit at $s \to s_0$, we obtain, taking into account (25) and (18),

$$\frac{v_k'^2(s_0)}{P_a'^2} = -\frac{v_k'^2(s_0)}{P_b'P_a'} - \frac{2y'(s_0)}{RP_a'}. \tag{26}$$

From (25) and (26) we obtain the equalities

$$P_b' + P_a' = \frac{2Ry'(s_0)}{(N\boldsymbol{y}, \boldsymbol{y})}, \; P_b'P_a' = -\frac{R^2 v_k'^2(s_0)}{(N\boldsymbol{y}, \boldsymbol{y})}, \tag{27}$$

which make it possible to determine P_b' and P_a'. They replace relation (20) in the case of a multiple CRF P when it loses its meaning due to the non-uniqueness of $\boldsymbol{y}$. We repeat that in relations (27) one should use the form $\boldsymbol{y}$ orthogonal to all BMs $\boldsymbol{v}_k$ of the system S corresponding to P_k, $(N\boldsymbol{y}, \boldsymbol{v}_k) = 0$. They take the simplest form if we accept the normalization condition $(N\boldsymbol{y}, \boldsymbol{y}) = 1$. In this case, P_b' and P_a' are defined by the expressions $R\left[y'(s_0) \pm \sqrt{y'^2(s_0) + v_k'^2(s_0)} \right]$.

Remark. The reasoning and conclusions made above regarding the CRFs of the system S equal to or different from the roots of equation (15) remain valid even in the case of their multiplicity in the system S. In this case, in all relations, starting from (22), one should write $\sum v_k^2(s)$ instead of $v_k^2(s)$ and $\sum v_k'^2(s)$ instead of $v_k'^2(s)$, where the sums apply to all BMs corresponding to a multiple CRF.

The next section demonstrates applications of some of the results obtained.

2. Maximum increase of the stability of an elastically supported two-span rod.

Further, as a system S, we consider an elastic rectilinear rod with a length equal to ℓ, of an arbitrary variable cross-section, freely supported at the ends on elastic supports with stiffness coefficients c_1 and c_2 accordingly, compressed by a longitudinal force constant along the length (Fig. 2 *a*).

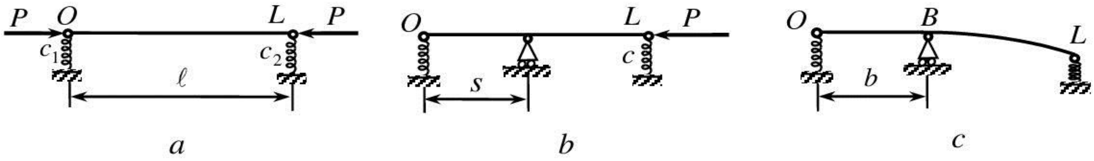

Fig. 2. Rods $OL(c_1,c_2)$ (*a*), $OL^{(1)}(s,c_1,c_2)$ (*b*) and semi-curved BM (*c*)

We are looking for such an optimal position of the intermediate rigid hinged support, at which the CRF of the rod reinforced with the intermediate support (Fig. 2 *b*) reaches its maximum value P_{MAX}.

The following notations are used:

$OL\left(c_1,c_2\right)$ – rectilinear elastic rod, the ends O and L of which are hinged on elastic supports with stiffness coefficients c_1 and c_2 respectively (Fig. 2 *a*);

$GH\left(c_1,c_2\right)$ – a rod formed from $OL\left(c_1,c_2\right)$ by removing segments OG and HL respectively from the left and right, and supported as $OL\left(c_1,c_2\right)$;

$GH^{(1)}\left(s,c_1,c_2\right)$ – a rod formed from $GH\left(c_1,c_2\right)$ by introducing an additional absolutely rigid hinged support at a distance s from the left end;

$GH_0^{(1)}\left(s,c_1,c_2\right)$ – a rod formed from $GH^{(1)}\left(s,c_1,c_2\right)$ by introducing a cut on an intermediate support.

$P_j[*]$ – j-th CRF of rod $*$.

In [8], the problem posed was solved for the particular case $c_1=\infty$. In this case, the desired optimal position and the corresponding BM depend on the value c_2 of the rigidity of the elastic support, and for some of its values, the maximum of CRF is realized at a special semi-curved BM, in which part of the rod remains straight and horizontal (Fig. 2 *c*). The conjugation point B of the horizontal and curved sections is determined by the equalities

$$c_2\cdot BL=c_2(\ell-b)=P_1\left[BL(\infty,\infty)\right],\ P_{MAX}=c_2(\ell-b). \tag{28}$$

Since an undeformed section remains to the left of the conjugation point when buckling along a semi-curved shape, it is possible to install or remove an arbitrary number of constraints on it that do not change this shape. The decrease of the rigidity of the left support from to is just such a removal, retaining the semi-curved BM and corresponding to it CRF, but possibly changing (increasing by 1) its number in the spectrum.

We designate A – the node of the 2nd BM of the rod $OL(\infty,\infty)$, supported at the ends on absolutely rigid supports, located at a distance $a=OA$ from the left support.

The spectrum of CRFs and the corresponding BMs of the rod $OL\left(c_1,c_2\right)$ contains all CRFs P_1, P_2, ... and BMs of the rod $OL(\infty,\infty)$ supported at the ends on absolutely rigid supports, and, in addition to them, one special CRF P^*, which corresponds to a rectilinear BM with a node A^* located at a distance a^* from the left support, at that

$$P^*=\frac{\ell}{1/c_1+1/c_2},\ a^*=\frac{c_2\ell}{c_1+c_2}. \tag{29}$$

These relations, as well as their inversion

$$c_1=P^*/a^*,\ c_2=P^*/\ell-a^*, \tag{30}$$

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allow us to consider P^* and a^* as parameters that characterize the elastic fixing of the rod as fully as the coefficients c_1 and c_2, and in the mathematical sense are a change of variables.

Let's agree to be located so that the node A^* of a special (rectilinear) BM is located not to the right of A, i.e., so that $a^* \le a$ is always satisfied.
We introduce the notation

$$P_M = P_2 \frac{\ell - a^*}{\ell - a}\ ,\ P_M \ge P_2 . \tag{31}$$

When looking for P_{MAX}, we consider the following cases.

Case 1. $P^* > P_M$. Let's place an intermediate support in the node A and consider the rod $OL_0^{(1)}(a, c_1, c_2)$. Its spectrum consists of the spectra of its two parts $OA(c_1, \infty)$ and $AL(\infty, c_2)$, each of which contains a force P_2 and one special CRF $c_1 a$ and $c_2(\ell - a)$, in addition

$$c_1 a \ge c_1 a^* = P^* > P_M \ge P_2 , \tag{32}$$

$$c_2\left(\ell - a\right) = P^* \frac{\ell - a}{\ell - a^*} = P^* \frac{P_2}{P_M} > P_2 , \tag{33}$$

i.e. both special CRFs are greater than P_2, whence it follows that after the imposition of a constraint that eliminates the cut, the force P_2 (which was 2-multiple and main in the spectrum of the rod $OL_0^{(1)}(a, c_1, c_2)$) will remain CRF of the rod $OL^{(1)}(a, c_1, c_2)$, i.e. $P_1^{(1)} = P_2 = P_{MAX}$ (by virtue of (16)) when placing the support in the node A of the second BM of the rod $OL(\infty, \infty)$. Other positions of the support (other than A) are not nodes of the second BM of the rod $OL(c_1, c_2)$ and, by virtue of statement A (Sec. 1.5), cannot provide the maximum critical force P_2.

Case 2. $P^* = P_M$. Inequality (32) remains valid, i.e. $c_1 a \ge P^* \ge P_2$, and in (33) the sign ">" changes to "=". P_2 remains the main CRF of the cut rod $OL_0^{(1)}(a, c_1, c_2)$, and at least 3-multiple. After the cut is eliminated, it will be at least 2-multiple the main CRF in the spectrum of the rod $OL^{(1)}(a, c_1, c_2)$, i.e. the optimal location of the intermediate support is the same as in case 1, and $P_{\mathrm{MAX}} = P_2$.

Case 3. $P_1 < P^* < P_M$. The left inequality means that $c_2 \ell \ge c_2\left(\ell - a^*\right) = P^* > P_1$. The right one leads to $c_2\left(\ell - a^*\right) < P_2\left(\ell - a^* / \ell - a\right) \Rightarrow c_2 < P_2 / \left(\ell - a\right)$. Both inequalities lead to the conclusion that there is a unique solution to the equation

$$c_2(\ell - x) = P_1\left[XL(\infty, \infty)\right], \tag{34}$$

where X is the cross-section of the rod at a distance $x = OX$ from its left end. This can be seen from the fact that in (34) the left side increases, and the right side decreases at x decreasing, i.e. when moving the cross-section X to the left. This solution $x = b$ satisfies the inequality $0 < b < a$. It was established in [8] that when an intermediate support is installed in a position B at a distance b from the left support, there is a semi-curved BM (Fig. 2 *c*), which corresponds to CRF P_B equal to

$$P_B = c_2 \cdot BL = c_2(\ell - b) = P_1\left[BL(\infty, \infty)\right] < P_1\left[AL(\infty, \infty)\right] = P_2 . \tag{35}$$

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This BM has a zero slope on the support B, which is a necessary condition for the extremum of the corresponding CRF (see (21)). It was proved in [8] that for other positions of the intermediate support, the slope of the main BM on the support cannot be zero. Since $P_B = P_1\left[BL(\infty,\infty)\right]$ is the CRF of the rod, shorter than $OL(\infty,\infty)$, the inequality $P_B > P_1 = P_1\left[OL(\infty,\infty)\right]$ is fulfilled, from which the estimates $P_B > P_1\left[OL(c_1,\infty)\right]$ and $P_B > P_1\left[OL(\infty,c_2)\right]$ follow, whence it follows that P_B cannot be the minimum of the main CRF of the rod $OL^{(1)}(s,c_1,c_2)$ as a function of the position s of the intermediate support. In addition to B, the node A^* of the rectilinear BM of the rod $OL(c_1,c_2)$ also satisfies the extremum condition for $P_1\left[OL^{(1)}(s,c_1,c_2)\right]$, since when the support is placed in A^*, its reaction at buckling in this BM is zero. Depending on the relative position of the sections A^* and B, the following relations hold:

$$b > a^* \Rightarrow P_B = c_2(\ell - b) < c_2(\ell - a^*) \Rightarrow P_B < P^* \Rightarrow P_B \text{ is the main CRF of } OL^{(1)}(b,c_1,c_2), \quad (36)$$

$$b = a^* \Rightarrow P_B = c_2(\ell - b) = c_2(\ell - a^*) \Rightarrow P_B = P^* \text{ is the main multiple CRF of } OL^{(1)}(b,c_1,c_2), \quad (37)$$

$$b < a^* \Rightarrow P_B = c_2(\ell - b) > c_2(\ell - a^*) \Rightarrow P_B > P^* \Rightarrow P_B \text{ is not the main CRF of } OL^{(1)}(b,c_1,c_2). \quad (38)$$

In combination with conditions $P_1 < P_B < P_2$, relations (36) show that when the point B of conjugation of the semi-curved BM is located between A and A^* this point provides the maximum CRF equal to $P_{MAX} = P_B$.

If B and A^* coincide, according to (37) and (35) $P^* = P_B < P_2$ and there are two linearly independent BMs – semi-curved and rectilinear, corresponding to CRF equal to $P_{MAX} = P_B = P^*$.

If B is to the left of A^*, then, as can be seen from (38), P_B will be greater than P^*, which is the second in the spectrum of $OL(c_1,c_2)$. Therefore, P_B cannot be the main CRF in $OL^{(1)}(s,c_1,c_2)$. Let's place a support in A^* and consider a cut rod $OL_0^{(1)}(a^*,c_1,c_2)$. Its left segment OA^* is shorter than OA. Therefore, $P_1\left[OA^*(\infty,\infty)\right] > P_1\left[OA(\infty,\infty)\right] = P_2 > P_B > P^*$. The right segment A^*L is shorter than BL, whence $P_1\left[A^*L(\infty,\infty)\right] > P_1\left[BL(\infty,\infty)\right] = P_B$. Thus, the cut rod $OL_0^{(1)}(a^*,c_1,c_2)$ has a 2-multiple main CRF equal to P^*, which corresponds to two BMs, with inclined straight sections OA^* and A^*L. After eliminating the cut in A^*, a rod $OL^{(1)}(a^*,c_1,c_2)$ is formed with the main CRF equal to P^*. Since it was second in the spectrum of $OL(c_1,c_2)$, $P_{\text{MAX}} = P^*$. There can be no other optimal positions of the intermediate support, since the necessary condition (21) of the extremum of CRF is not satisfied anywhere else.

Case 4. $P^* = P_1$. CRF $P^* = P_1$ is 2-multiple in the spectrum of rod $OL(c_1,c_2)$ and, by virtue of (16), after the introduction of a support at any point s of the rod, CRF of rod $OL^{(1)}(s,c_1,c_2)$ is equal to $P_{MAX} = P_1^{(1)} = P_1 = P^*$. The corresponding BM is a linear combination of the BM $v_1(x)$ of rod $OL(\infty,\infty)$ and the rectilinear BM of rod $OL(c_1,c_2)$ and can be expressed explicitly up to a constant factor

$$y(x) = (s - a^*)v_1(x) - v_1(s)(x - a^*). \quad (39)$$

Case 5. $(P_1/2) \le P^* < P_1$. Due to (16), the desired maximum $P_{MAX} \le P_1$. The condition

$P^* \geq (P_1/2)$ implies the inequality

$$\frac{1}{P^*} = \frac{1}{c_1\ell} + \frac{1}{c_2\ell} \leq \frac{2}{P_1}, \tag{40}$$

from which it follows that at least one of the numbers $c_1\ell$, $c_2\ell$ is greater than or equal to P_1. These numbers are the special CRFs of rods $OL(c_1,\infty)$ and $OL(\infty,c_2)$ formed from $OL(c_1,c_2)$ by setting a rigid support in L and O, respectively. The spectrum of each of them consists of the spectrum of the rod $OL(\infty,\infty)$ and one of the special CRFs, which corresponds to a rectilinear BM. The optimal position of the movable support is that of the two points L and O, which provides the value of the special CRF greater than or equal to P_1, or both of these points, if each of the numbers $c_1\ell$, $c_2\ell$ is not less than P_1. In this case, $P_{MAX} = P_1$ is reached, which corresponds to the main BM of the rod $OL(\infty,\infty)$ (and, possibly, a special one, if one of the numbers $c_1\ell$, $c_2\ell$ is equal to P_1). There are no other optimal positions, because other positions are not nodes of this BM.

Case 6. $P^* < (P_1/2)$. If one of the numbers $c_1\ell, c_2\ell$ exceeds or equals P_1, all the conclusions of case 5 remain valid, in particular, $P_{MAX} = P_1$ when installing a support at one of the ends of the rod. Otherwise, consider a cut rod $OL_0^{(1)}(s,c_1,c_2)$ with an arbitrary location of the intermediate support at a distance s from the support O. It has two CRFs $c_1 s$ and $c_2(\ell - s)$, which correspond to special BMs, in which one of the segments to the right or left of the support rotates, remaining straight. Each of these CRFs, due to the inequalities $c_1 s < c_1\ell$, $c_2(\ell - s) < c_2\ell$ is less than the largest of the numbers $c_1\ell$, $c_2\ell$, less than P_1. The remaining CRFs are CRFs of rods that are shorter than $OL(\infty,\infty)$, and therefore they exceed P_1. Thus, $c_1 s$ and $c_2(\ell - s)$ are the lowest CRFs in the spectrum of the rod $OL_0^{(1)}(s,c_1,c_2)$. After the cut is eliminated, a rod $OL^{(1)}(s,c_1,c_2)$ is formed whose main CRF does not exceed the value of the highest of the numbers $c_1 s < c_1\ell$, $c_2(\ell - s) < c_2\ell$. At the same time, the values $c_1\ell$ and $c_2\ell$ realise when the support is installed at the right and left ends of the rod, respectively. Thus, in the considered case $P_{MAX} = \max\{c_1\ell, c_2\ell\}$, and the optimal position of the support is the right end L of the rod, if $c_1 > c_2$, the left end O, if $c_1 > c_2$, and any of them, if $c_1 = c_2$.

5 RESEARCH RESULTS DISCUSSION

The presented results make it possible to study the stability of a wide class of rod systems, including mechanisms. They show that even with such an extended approach, against the background of a multiplicity of equilibrium positions, one can speak of a discrete spectrum of critical forces and buckling modes in the traditional sense, which makes it possible to apply the expansion of system configurations by these modes. Note that the introduction of a constraint significantly changes the critical forces and buckling modes only if the constraint is orthogonal to all special forms. In the case of a constraint in the form of a concentrated hinge support, this orthogonality can only be ensured when it is installed in some parts of the system. In most cases, in practice, there are systems that do not have special forms. For them, the results of the work can be applied without limitations.

6 CONCLUSIONS

In the work, the influence of the introduction of a constraint on the stability of rod systems is studied. The study made it possible to draw a number of qualitative conclusions regarding the results of such reinforcement. Based on them, simple qualitative features of optimal locations for imposed constraint are formulated that provide the maximum critical force of the enhanced system. This makes it possible in many cases to determine these positions practically without calculations, which is demonstrated by the example of a rod hinged at the ends on elastic supports and reinforced with an intermediate hinged support. Note that for certain values of the stiffness coefficients of the end supports, the optimal rod buckles at loss of stability in a special semi-curved mode, in which one of the spans remains straight. Although special attention is paid to the constraint in the form of a concentrated hinge support, the results obtained allow us to consider generalized constraints with an arbitrary spatial distribution of reactive forces. Corresponding generalizations will be the subject of further research.

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