UDC 531.381

EVOLUTION OF ROTATIONAL MOTIONS IN A RESISTIVE MEDIUM OF A NEARLY DYNAMICALLY SPHERICAL GYROSTAT SUBJECTED TO CONSTANT BODY-FIXED TORQUES

D. Leshchenko[1], T. Kozachenko[1]
[1]*Odesa state academy of civil engineering and architecture*

Abstract. A satellite or a spacecraft in its motion about the center of mass is affected by the torques of forces of various physical nature. It is influenced by the gravitational, aerodynamic torques, the torques due to the light pressure, and the torques due to the motions of masses inside the body. These motions may have various causes, for example, the presence of fluid in the cavities in the body (for example, liquid fuel or oxidizer in the tanks of a rocket). Therefore, there is a necessity to study the problems of the dynamics of bodies with cavities containing a viscous fluid, to calculate the motion of spacecrafts about the center of mass, as well as their orientation and stabilization. The mentioned torques, acting on the body, are often relatively small and can be considered as perturbations. It is natural to use the methods of small parameter to analyze the dynamics of rigid body under the action of applied torques. The method applied in this paper is the Krylov-Bogolubov asymptotic averaging method.

The studies of F. L. Chernousko showed that solving the problems of dynamics of a rigid body with a viscous fluid can be subdivided into two parts – the hydrodynamic and dynamic ones – which can greatly simplify the initial problem.

We investigated the motion about its center of mass in a resistive medium of a nearly dynamically spherical rigid body with a cavity filled with a viscous fluid at small Reynolds numbers, subjected to constant body-fixed torque which is described by the system of differential equations, considering the asymptotic approximation of the moments of the viscous fluid in the cavity. The determination of the motions of forces acting on the body from side of the viscous fluid in the cavity was proposed in the works of F. L. Chernousko. We obtained the system of equations of motion in the standard form which refined in square-approximation by small parameter. The Cauchy problem for a system determined after averaging was analyzed. The evolution of the motion of a rigid body under the action of small internal and external torques of forces is described by the solutions which obtained as a result of asymptotic, analytical and numerical calculations over an infinite time interval.

Keywords: nearly dynamically spherical rigid body, cavity, viscous fluid, constant torque, resistive medium.

ЕВОЛЮЦІЯ ОБЕРТАЛЬНИХ РУХІВ В СЕРЕДОВИЩІ З ОПОРОМ, БЛИЗЬКОГО ДО ДИНАМІЧНО СФЕРИЧНОГО ГІРОСТАТА ПІД ДІЄЮ ПОСТІЙНИХ МОМЕНТІВ В ЗВ'ЯЗАНИХ З ТІЛОМ ОСЯХ

Лещенко Д. Д.[1], Козаченко Т. О.[1]
[1]*Одеська державна академія будівництва та архітектури*

Анотація. На супутник або космічний апарат у своєму русі відносно центра мас діють моменти сил різної фізичної природи. Це гравітаційні, аеродинамічні моменти, а також моменти, обумовлені рухом деяких мас в тілі. Такі рухи можуть бути викликані різними причинами: наприклад, наявністю рідини в порожнинах в тілі (наприклад, рідке паливо або окислювач в резервуарах ракети). Таким чином, є необхідність вивчення задач динаміки твердих тіл з в’язкою рідиною для дослідження руху космічних апаратів навколо центра мас, а також їх орієнтації та стабілізації. Вказані моменти, що діють на тіло відносно малі і можуть розглядатися як збурення. Для аналізу динаміки твердого тіла під дією прикладених моментів використовують методи малого параметра. Метод, який застосовано у цій роботі – метод усереднення Крилова-Боголюбова.

Задачі динаміки твердого тіла з порожнинами, що містять в’язку рідину, представляють значно більші труднощі, ніж у випадку ідеальної рідини. В роботах Ф. Л. Черноуська показано, що розв’язування задач динаміки твердого тіла з однорідною в’язкою рідиною можна розкласти на дві частини – гідродинамічну та динамічну – що дозволяє спростити початкову задачу. Асимптотичний розв’язок був одержаний для опису еволюції твердого тіла з порожниною, заповненою рідиною великої в’язкості, на великому проміжку часу.

Розглядається рух відносно центра мас в середовищі з опором близького до динамічно сферичного твердого тіла з порожниною, заповненою в’язкою рідиною при малих числах Рейнольдса під дією постійного моменту в зв’язаних з тілом осях, який описується системою диференційних рівнянь з урахуванням в асимптотичному наближенні моментів сил в’язкої рідини в порожнині тіла. Визначення моментів сил, що діють на тіло зі сторони в’язкої рідини в порожнині, було запропоновано в роботах Ф. Л. Черноуська. Отримано систему рівнянь руху в стандартній формі, уточнену в квадратичному наближенні за малим параметром. Проаналізовано задачу Коші для системи, визначеної після усереднення. Еволюція руху твердого тіла під дією таких внутрішніх і зовнішніх моментів сил описується розв’язками, отриманими в результаті асимптотичних, аналітичних і чисельних розрахунків на нескінченному інтервалі часу.

Ключові слова: близьке до динамічно сферичного тверде тіло, порожнина, в’язка рідина, сталий момент, середовище з опором.

1 INTRODUCTION

The problems of the dynamics of rigid bodies with cavities containing fluid are among the classical problems of mechanics: a fundamental study of the rotational motion of a rigid body having a cavity, filled with a homogeneous ideal fluid, was carried out by N. Ye. Zhukovskii [1]. The interest to the problems of the rotation of rigid bodies with fluid-containing cavity has arisen in connection with the development of the rocket and space technology as well as geophysical and astrophysical applications.

2 ANALYSIS OF LITERATURE DATA AND RESOLVING THE STUDY

The presentation of the results on the dynamics of rigid body motion about the center of mass with fluid-filled cavities is given in books by N. N. Moiseyev and V. V. Rumyantsev [2], and I. A. Lukovsky [3].

The problems of rigid body dynamics with cavities containing a viscous fluid are significantly more difficult that in the case of ideal fluid. An important contribution to the solution of these problems has been made by the works of F. L. Chernousko et al. [4, 5]. These studies showed that solving the problems of the dynamics of the rigid body with viscous fluid in cavity can be separated into two parts: the hydrodynamic and dynamic ones, which represents a considerable simplification of the original problem. An asymptotic solution was obtained describing the evolution of the motion of a body having a cavity with a fluid of high viscosity over a long-time interval.

In the paper [6], the initial period of rotational motion of a body with a cavity containing a fluid of high viscosity was investigated. The article [7] is devoted to studying the stabilizing effect of a viscous fluid in a cavity on the rotation of a top around the given axes. In [8], the oscillations on an elliptic orbit of a satellite with moments of inertia close to one another and a spherical cavity filled with a viscous fluid are studied. In papers [9–11] the fast rotational motions about the center of mass of a dynamically asymmetric satellite with a cavity filled with viscous fluid under the action of gravitational and light pressure torques, and medium resistance were investigated.

In [12], the inertial motion of a rigid body with a spherical or ellipsoidal cavity filled with a viscous fluid was studied by the asymptotic method. A numerical analysis of the change in the vector of the moment of momentum of a rigid body with a cavity filled with a viscous fluid was performed in [13]. The article [14] presents the analytical and numerical results obtained in the study of systems containing a rigid body with a cavity filled with a viscous fluid. In [15], an approach was proposed for modeling the dynamics of a rigid body with cavity filled with a high-viscosity fluid.

The problem of deceleration of rotations of a dynamically symmetric body with a cavity filled with lightly viscous fluid was investigated in [16]. In paper [17], the problem of time-optimal deceleration of a dynamically asymmetric body with cavity filled with viscous fluid in a resistive medium was studied.

The papers [18, 19] study the motion about the center of mass of a nonsymmetric rigid body influenced by two small perturbation torques: a constant one in the body-fixed axes and a linear dissipative one or, alternatively, a constant one and a torque involving the terms quadratically depending on the angular velocity.

In the works [20–22] analytical solutions are obtained for the problem of a rigid body close to symmetrical one, as well as of a body with arbitrary inertia characteristics by a torque which is constant in the body-fixed axes.

In paper [23], the analytic solution has been introduced for the rotation of a rigid body having spherical ellipsoid of inertia subjected to a constant torque.

The motion of a close to dynamically spherical rigid body with a cavity filled with a viscous fluid at low Reynolds number was investigated in [24]. Qualitative and quantitative results of motion in a resistive medium of a nearly dynamically spherical rigid body with a cavity containing fluid of high viscosity was studied in [25]. In paper [26], the motion about the center of mass of a nearly dynamically spherical rigid body with a cavity filled with a fluid of high viscosity and subjected to constant body-fixed torques was considered. In work [27] the case of a rigid body motion investigated in [24] was extended in the presence of the third component of the gyrostatic moment.

Consider the motion in space of a rigid body with a spherical cavity filled with a fluid of high viscosity relative to the center of inertia.

We assume that the torques which is constant in the body-connected axes have the form

$$M_i^c = \varepsilon^2 M_i = \text{const},\ i = 1,2,3, \tag{1}$$

where $0 < \varepsilon << 1$ is a small parameter.

We assume that the torque of the resistant forces is proportional to the angular momentum of the body with a "frozen" fluid [4, 11, 16, 17, 28]

$$\mathbf{M}^r = -\varepsilon^2 \lambda \mathbf{J}\boldsymbol{\omega}, \tag{2}$$

where λ is a positive coefficient of proportionality that depends on the properties of the medium and on the shape of the body and $\mathbf{J} = diag(A, B, C)$ is the tensor of inertia of the body with a fluid in the cavity, $0 < \varepsilon \ll 1$ is a small parameter.

We write the equations of motion for the system under consideration in projections onto the principal central axes of inertia by [4, 5]

$$\begin{gathered} A\frac{dp}{dt} + (C - B)qr = \varepsilon^2 M_1 - \varepsilon^2 \lambda Ap + \\ + \frac{\rho P_0}{\nu ABC} p\left[C(A - C)(A + C - B)r^2 + B(A - B)(A + B - C)q^2\right]. \end{gathered} \tag{3}$$

Here, A, B, and C are the principal central moments of inertia of the system, $p,\ q,\ r$ denote the projections of the absolute angular velocity $\boldsymbol{\omega}$ onto the principal central axes of inertia, ρ is the density of the fluid, and ν is the kinematic coefficient of viscosity. The first expression on the right-hand side of (3) defines, in the asymptotic approximation, the torque of forces of a viscous fluid in the cavity of the body [4, 5], $P_0 > 0$ is the scalar coefficient depending on the shape of the cavity. In the case of a spherical cavity of radius b, we have according to [4, 5]

$$P_0 = \frac{8\pi b^7}{525}. \tag{4}$$

The other equations are obtained from (3) by cyclic permutation of symbols $A,\ B,\ C$ and $p,\ q,\ r$.

The Reynolds number is assumed small: $\text{Re} = l^2 T_*^{-1} \nu^{-1} << 1$ [4, 5]. Here l is a characteristic linear dimension of the cavity, T_* is a characteristic time scale of the relative motion, which inversely proportional to the characteristic angular velocity ω. If l and T_* are taken as the units of measurement of length and time then following [4, 5] the kinematic coefficient of viscosity of the fluid is a large parameter $\nu = 1/\text{Re} >> 1$ and $\nu^{-1} << 1$. We assume that the nondimensional quantity $\nu^{-1} \sim \varepsilon$.

3 PURPOSE AND TASKS THE STUDY

Consider the case of a nearly dynamically spherical rigid body, when the principal central moments of inertia of a "frozen" rigid body are close to one another and represent them in the form

$$A = J_0 + \varepsilon A', \;\; B = J_0 + \varepsilon B', \;\; C = J_0 , \tag{5}$$

where $0 < \varepsilon << 1$ is a small parameter of the same order just as in (1) – (3). If $\varepsilon = 0$ equations of motion (3) describe the motion of a spherically symmetric rigid body. Assume also that there are estimates

$$|A - B| = O(\varepsilon^2 J_*), \quad |A' - B'| = O(\varepsilon J_*), \quad J_* \sim J_0 . \tag{6}$$

Then, following (2.4), (2.5), the expressions hold

$$A - B = \varepsilon(A' - B') = \varepsilon^2 J_*, \quad A - C = \varepsilon A', \quad B - C = \varepsilon B' . \tag{7}$$

After the transition to slow time $\tau = \varepsilon t$ and transformations of system (3), taking into account relations (5) – (7), we obtain a perturbed Euler system of the form (terms of order ε^2 and higher are discarded):

$$\begin{aligned}
\frac{dp}{d\tau} &= \frac{B'}{J_0}\left(1 - \varepsilon \frac{A'}{J_0}\right) qr + \varepsilon f_{1p}(p,q,r), \qquad p(0) = p_0, \\
\frac{dq}{d\tau} &= -\frac{A'}{J_0}\left(1 - \varepsilon \frac{B'}{J_0}\right) pr + \varepsilon f_{1q}(p,q,r), \quad q(0) = q_0, \\
\frac{dr}{d\tau} &= \frac{A' - B'}{J_0} pq + \varepsilon f_{1r}(p,q,r), \qquad r(0) = r_0.
\end{aligned} \tag{8}$$

Here, r is the slow variable in slow time τ. The system of differential equations (8) is an essentially nonlinear system in which the frequency depends on the slow variable r. In (8) perturbations were introduced

$$\begin{aligned}
\varepsilon f_{1p}(p,q,r) &= mp\left\{A'\left[J_0 - \varepsilon(A' + 2B')\right]r^2 + (A' - B')\left[J_0 - \varepsilon(A' - B')\right]q^2\right\} + \\
&\quad + \varepsilon \frac{M_1}{J_0}\left(1 - \varepsilon \frac{A'}{J_0}\right) - \varepsilon\lambda p, \\
\varepsilon f_{1q}(p,q,r) &= mq\left\{B'\left[J_0 - \varepsilon(2A' + B')\right]r^2 - (A' - B')\left[J_0 + \varepsilon(A' - B')\right]p^2\right\} + \\
&\quad + \varepsilon \frac{M_2}{J_0}\left(1 - \varepsilon \frac{B'}{J_0}\right) - \varepsilon\lambda q, \\
\varepsilon f_{1r}(p,q,r) &= -mr\left\{B'\left[J_0 - \varepsilon(2A' - B')\right]q^2 + A'\left[J_0 - \varepsilon(2B' - A')\right]p^2\right\} + \\
&\quad + \varepsilon \frac{M_3}{J_0} - \varepsilon\lambda r,
\end{aligned} \tag{9}$$

where, $m = \rho P_0 / \nu J_0^3$. The torque of the influence of the viscous fluid in the rigid body cavity is small [4, 5].

The solution of the system (8) for $\varepsilon = 0, \; \nu^{-1} = 0$ has a form

$$p = a\cos\varphi, \quad q = -\frac{J_0 a w \sin\varphi}{B' r}, \quad r = r_0. \tag{10}$$

Here, $a = \sqrt{p_0^2 + (\dot p_0/w)^2}$ is the amplitude (slow variable), $\varphi = w\tau + \varphi_0$ is the phase, $w = r\sqrt{A'B'}/J_0$, $A'B' > 0$, φ_0 is the initial phase, $\cos\varphi_0 = p_0/a$, $\sin\varphi_0 = -q_0\sqrt{B'/A'}/a$ by assumption.

We pass from the slow variables p, q, r to the new slow variables a, r and the phase φ. We use the change of variables for this:

$$p = a\cos\varphi, \quad q = -\frac{J_0 a w \sin\varphi}{B' r}, \quad r = r. \tag{11}$$

We differentiate expressions (11) by virtue of perturbed system. After series of transformations, we get the system in standard form

$$\begin{gathered}
\dot a\cos\varphi - a\dot\varphi\sin\varphi = -a\,w(r)\sin\varphi + \varepsilon f_{2p},\\
\dot a\sin\varphi + a\dot\varphi\cos\varphi = a\,w(r)\cos\varphi - \sqrt{\frac{B'}{A'}}\varepsilon f_{2q},\\
\dot r = \frac{B'-A'}{J_0}a^2\sqrt{\frac{A'}{B'}}\sin\varphi\cos\varphi + \varepsilon f_{2r}, \quad w(r) = \frac{r}{J_0}\sqrt{A'B'},\\
\varepsilon f_{2p} = \varepsilon\frac{aA'}{J_0}w(r)\sin\varphi + \varepsilon\frac{M_1}{J_0}(1-\varepsilon\frac{A'}{J_0}) - \varepsilon\lambda a\cos\varphi +\\
+ma\cos\varphi\left\{A'\left[J_0 - \varepsilon(A'+2B')\right]r^2 + (A'-B')\left[J_0 - \varepsilon(A'-B')\right]a^2\frac{A'}{B'}\sin^2\varphi\right\},\\
\varepsilon f_{2q} = \varepsilon\frac{a\sqrt{A'B'}}{J_0}w(r)\cos\varphi + \varepsilon\frac{M_2}{J_0}(1-\varepsilon\frac{B'}{J_0}) + \varepsilon\lambda a\sqrt{\frac{A'}{B'}}\sin\varphi -\\
-ma\sqrt{\frac{A'}{B'}}\sin\varphi\left\{B'\left[J_0 - \varepsilon(2A'+B')\right]r^2 - (A'-B')\left[J_0 + \varepsilon(A'-B')\right]a^2\cos^2\varphi\right\},\\
\varepsilon f_{2r} = -ma^2 rA'\left\{J_0 - \varepsilon\left[(2A'-B')\sin^2\varphi + (2B'-A')\cos^2\varphi\right]\right\} + \varepsilon\frac{M_3}{J_0} - \varepsilon\lambda r.
\end{gathered} \tag{12}$$

We solve equations (12) with respect to $\dot a$ and $\dot\varphi$, and get a system

$$\begin{gathered}
\dot a = \varepsilon f_{2p}\cos\varphi - \varepsilon f_{2q}\sqrt{\frac{B'}{A'}}\sin\varphi,\\
\dot\varphi = w(r) - \frac{1}{a}\varepsilon f_{2p}\sin\varphi - \frac{1}{a}\varepsilon f_{2q}\sqrt{\frac{B'}{A'}}\cos\varphi.
\end{gathered} \tag{13}$$

We substitute (11) into the third equation (8) for the variable r. Taking into account the change of variables and standard transformations we obtain the following system of equations:

$$\begin{gathered}
\dot a = \varepsilon\frac{(A'-B')}{J_0}aw(r)\sin\varphi\cos\varphi + \frac{\varepsilon}{J_0}\left[M_1(1-\varepsilon\frac{A'}{J_0})\cos\varphi - M_2(1-\varepsilon\frac{B'}{J_0})\sqrt{\frac{B'}{A'}}\sin\varphi\right] -\\
-\varepsilon\lambda a + ma\left\{r^2\left[A'\cos^2\varphi(J_0-\varepsilon A') - 2\varepsilon A'B' + B'\sin^2\varphi(J_0-\varepsilon B')\right] +\right.
\end{gathered}$$

$$
\begin{aligned}
&+a^2\frac{(A'-B')^2}{B'}\left[J_0-\varepsilon(A'+B')\right]\sin^2\varphi\cos^2\varphi\bigg\},\\
&\dot r=\frac{B'-A'}{J_0}a^2\sqrt{\frac{A'}{B'}}\sin\varphi\cos\varphi+\varepsilon\frac{M_3}{J_0}-\varepsilon\lambda r-\\
&-ma^2rA'\left\{J_0-\varepsilon\left[(2A'-B')\sin^2\varphi+(2B'-A')\cos^2\varphi\right]\right\},\\
&\dot\varphi=w(r)-\varepsilon\frac{w(r)}{J_0}(A'\sin^2\varphi+B'\cos^2\varphi)-m(A'-B')\sin\varphi\cos\varphi\left\{r^2\left[J_0-\varepsilon(A'+B')\right]+\right. \quad (14)\\
&\left.+a^2\left[(J_0+\varepsilon(A'-B'))\cos^2\varphi+(J_0-\varepsilon(A'-B'))\frac{A'}{B'}\sin^2\varphi\right]\right\}-\\
&-\frac{\varepsilon}{aJ_0}\left[M_1(1-\varepsilon\frac{A'}{J_0})\sin\varphi+M_2(1-\varepsilon\frac{B'}{J_0})\sqrt{\frac{B'}{A'}}\cos\varphi\right].
\end{aligned}
$$

Here, the quantity $w(r)=r\sqrt{A'B'}/J_0$ has the meaning of the perturbed frequency of the transformed system. After the averaging of the system (14) over the phase φ [29] we find:

$$
\begin{aligned}
&\dot a=ca+ma(\beta r^2+\alpha a^2),\\
&\dot r=cr-m\gamma a^2r+\varepsilon\frac{M_3}{J_0}.
\end{aligned}
\qquad (15)
$$

Here we have introduced the notations

$$
\begin{aligned}
&c=-\varepsilon\lambda,\quad \beta=\frac{1}{2}(A'+B')\left[J_0-\varepsilon(A'+B')\right]-\varepsilon A'B',\\
&\alpha=\frac{(A'-B')^2}{8B'}\left[J_0-\varepsilon(A'+B')\right],\quad \gamma=-A'\left[J_0-\frac{1}{2}\varepsilon(A'+B')\right].
\end{aligned}
$$

We transform system (15) to the form:

$$
\begin{aligned}
&\dot x=2x(c+m\beta y+m\alpha x),\\
&\dot y=2(yc+m\gamma xy+\frac{\varepsilon M_3}{J_0}\sqrt{y}).
\end{aligned}
\qquad (16)
$$

The variables $x=a^2$, $y=r^2>0$ are introduced here, $r=\sqrt{y}$. Note that in system (16) x, y are slow variables.

It can be directly observed that, in the first approximation the equations for x and y in (16) include only constant in the body-connected axis torque M_3. The terms containing the perturbation torques M_1, M_2 drop out upon averaging.

4 BASIC RESULTS

System (16) was solved numerically with the initial conditions $x(0)=1$, $y(0)=1$ and task parameters $P_0=0.48$ m^7, $\nu=1000$ m^2/s, $\varepsilon=0.1$, $\rho=1260$ kg/m^3, $\lambda=1.25$ rad/s. We obtain

$$\begin{aligned}
&x(t)=1+(2c+1.21(\alpha+\beta))t+(c(2c+3.63(\alpha+\beta))+\\
&+\beta(0.13\alpha+0.73\gamma+0.12M_3)+(1.21\alpha+0.85\beta)^2)t^2+O(t^3),\\
&y(t)=1+(2c+1.21\gamma+0.2M_3)t+\\
&+(0.73\gamma(\alpha+\beta+\gamma+4.97c)+0.02M_3(9\gamma+c)+(1.41c+0.12M_3)^2)t^2+O(t^3).
\end{aligned} \quad (17)$$

The plots of the changing values a^2 and r^2 of the squared equatorial and axial component of the angular velocity vector of the rigid body are constructed and represented in two cases.

In the first case (Figs. 1, 2) $J_0=1,\ A'=5.1,\ B'=5$, in the second case (Figs. 3, 4) $J_0=3,\ A'=1.3,\ B'=1$.

5 DISCUSSION OF THE RESULTS OF THE STUDY

How we can see from the plots in both cases variable $y=r^2$ decreases on the interval of time $[0;10]$ and $[0;20]$ asymptotically approaching zero (Figs. 2, 4).

The variable $x=a^2$ decreases in case 1, asymptotically approaching zero (Fig. 1). In the second case in the presence of dissipation and a small constant torque $x=a^2$ decreases to zero (Fig. 3), and under the action of only the internal torque in the interval $[0;20]$ increases reaching the value $x=2$.

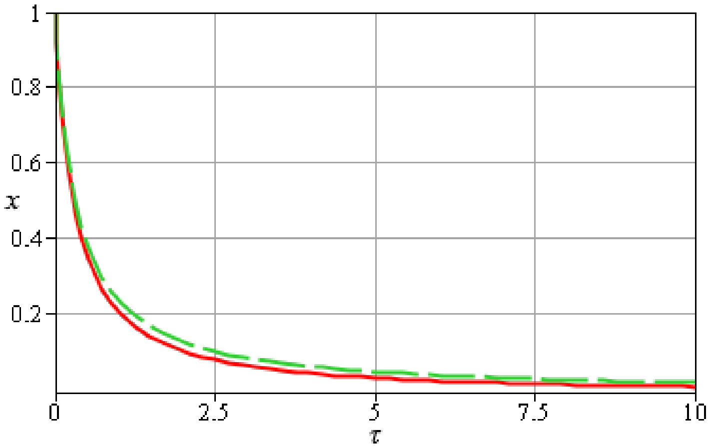

Fig. 1. Comparison of the graphs $x=a^2$ under the action of internal torque (– –) and (——) of both internal and small constant torque $M_3=-0.135$

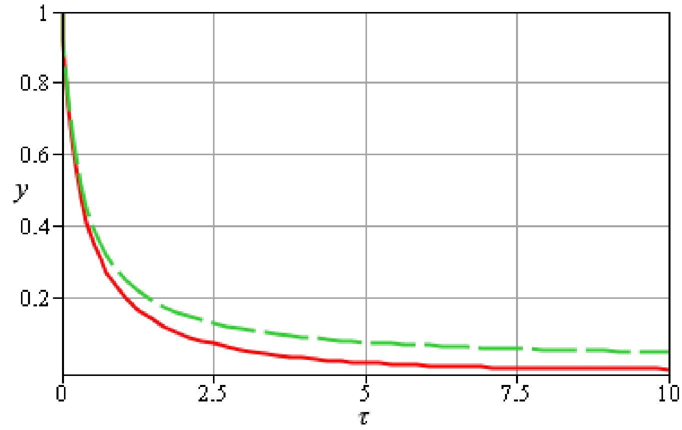

Fig. 2. Comparison of the graphs $y = r^2$ under the action of internal torque (— -) and (——) of both internal and small constant torque $M_3 = -0.135$

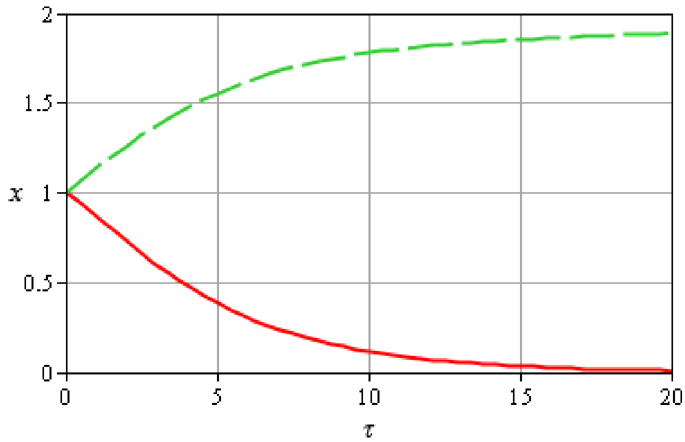

Fig. 3. The plot of variable $x = a^2$ under the action of internal torque (— -) and (——) of both internal and small constant torque $M_3 = -0.135$

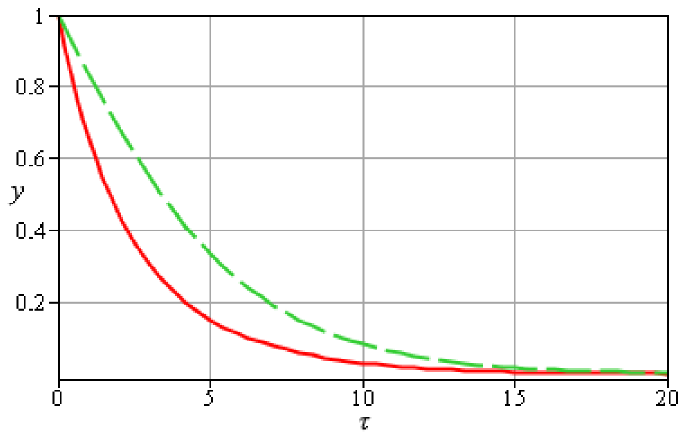

Fig. 4. The plot of variable $y = r^2$ under the action of internal torque (— -) and (——) of both internal and small constant torque $M_3 = -0.135$

6 CONCLUSIONS

The motion of a nearly dynamically spherical rigid body in a resistive medium with a cavity filled with viscous fluid at low Reynolds number under the action of constant body-fixed torques is investigated. A system of equations of motion in standard form which refined in the quadratic approximation by small parameter is obtained. The Cauchy problem for the system determined after averaging is analyzed. The evolution of the rigid body motion is described by solutions obtained asymptotically, and numerically.

Results summed up in this paper make it possible to analyze motions of artificial satellites and celestial bodies under the influence of small internal and external torques

References

1. Zhukovskii, N. Ye. (1948). O dvizhenii tverdogo tela, imeyushchiye polosti, napolnennyye odnorodnoyu kapelnoyu zhidkostyu [On the motion of a rigid body with cavities filled with a homogeneous liquid drop]. *Selected Works*. Moscow-Leningrad: Gostekhizdat. 1. 31–152. [in Russian].
2. Moiseyev, N. N., Rumyantsev, V. V. (1968). *Dynamic Stability of bodies Containing Fluid.* New York: Springer.
3. Lukovsky, I. A. (2015). *Mathematical Models for Rigid Bodies with a Liquid.* Walter de Gruyter GmbH Co KG.
4. Chernousko, F. L (1972). *The Movement of a Rigid Body with Cavities Containing a Viscous Fluid.* NASA. Washington.
5. Chernousko, F. L., Akulenko, L. D., Leshchenko, D. D. (2017). *Evolution of Motions of a Rigid Body About its Center of Mass*. Cham: Springer International Publishing. https://doi.org/10.1007/978-3-319-53928-7.
6. Kobrin, A. I. (1969). On the motion of a hollow body with viscous liquid about its center of mass in a potential body force field. *Journal of Applied Mathematics and Mechanics*. 33(3). 418–427.
7. Smirnova, E. P. (1974). Stabilization of free rotation of an asymmetric top with cavities completely filled with a fluid. *Journal of Applied Mathematics and Mechanics*. 38(6). 931–935.
8. Osipov, E. P., Sulikashvili, R. S. (1978). On oscillations of a rigid body with a cavity completely filled with a viscous liquid in an ellipsoidal orbit. *Tr. Tbilis. Matem. Inst. Akad. Nauk Gruz SSR* 58. 175–186. [in Russian].
9. Akulenko, L. D., Leshchenko, D. D., Rachinskaya, A. L. (2007). Evolution of rotations of a satellite with cavity filled with viscous fluid. *Mekh. Tverd. Tela* 37. 126–139. [in Russian].
10. Akulenko, L. D., Zinkevich, Y. S., Leshchenko, D. D., Rachinskaya, A. L. (2011). Rapid rotations of a satellite with a cavity filled with viscous fluid under the action of moments of gravity and light pressure forces. *Cosmic Research*. 49(5). 440–451. https://doi.org/10.1134/S0010952511050017.
11. Leshchenko, D., Akulenko, L., Rachinskaya, A., Shchetinina, Yu. (2015). Rotational motion of a satellite with viscous fluid under the action of the external resistance torque. *Mathematics in Engineering, Science and Aerospace*. 6(3). 383–391.
12. Baranova, E. U., Vil'ke, V. G. (2013). Evolution of motion of a rigid body with a fixed point and an ellipsoidal cavity filled with a viscous fluid. *Moscow Univ. Mech. Bull*. 68(1). 15–20.
13. Rachinskaya A. L. (2015). Motion of a solid body with cavity filled with viscous liquid. *Cosmic Research*. 53(6). 476–480.
14. Disser, K., Galdi, G.P., Mazzone, G., Zunino, P. (2016). Inertil motions of a rigid body with a cavity filled with a viscous liquid. *Archive for Rational Mechanics and Analysis*. 221(1). 487–526.
15. Ramodanov, S. M., Sidorenko, V. V. (2017). Dynamics of a rigid body with an ellipsoidal cavity filled with viscous fluid. *International Journal of Non-Linear Mechanics*. 95. 42–46. https://doi.org/10.1016/j.ijnonlinmec.2017.05.006.
16. Akulenko, L. D., Leshchenko, D. D., Rachinskaya, A. L. (2010). Optimal deceleration of rotation of a dynamically symmetric body with a cavity filled with viscous liquid in a resistive medium.

Journal of Computer and System Sciences International. 49(2). 222–226. https://doi.org/10.1134/S1064230710020073.
17. Akulenko, L. D., Leshchenko, D. D., Rachinskaya, A. L. (2012). Optimal deceleration of rotation of an asymmetric body with a cavity filled with viscous fluid in a resistive medium. *Journal of Computer and System Sciences International*. 51(1). 38–48.
18. Neishtadt, A. T. (1980). Evolution of rotation of a solid, acted upon by the sum of a constant and dissipative perturbing moments. *Mechanics of Solids*. 15(6). 21–27.
19. Pivovarov, M. L. (1985). The motion of a gyroscope with low self-excitation, *Izv. Akad. Nauk SSR. Mekh. Tverd. Tela*. 6. 23–27. [in Russian].
20. Van der Ha, J. C. (1985). Perturbation solution of attitude motion under body-fixed torques. *Acta Astronautica*. 12(10). 861–869.
21. Kane, T. R., Levinson, D. A. (1987). Approximate description of attitude motion of a torque-free, nearly axisymmetric rigid body. *Journal of the Astronautical Sciences*. 35(4). 435–446.
22. Ayobi, M. A., Longuski, J. M. (2008). Analytical solution for translational motion of spinning-up rigid bodies subject to constant body-fixed forces and moments. *Trans. ASME. Journal of Applied Mechanics*. 75(1). 011004/1-011004/8.
23. Romano, M. (2008). Exast analytic solution for a rotation of a rigid body having spherical ellipsoid of inertia and subjected to a constant torque. *Celestial Mechanics and Dynamical Astronomy*. 100. 181–189.
24. Akulenko, L. D., Leshchenko, D. D., Paly, K. S. (2021). Perturbed rotational motions of a spheroid with cavity filled with a viscous fluid. *Proc. IMechE Part C: Journal of Mechanical Engineering Science*. 235(20). 4833–4837. https://doi.org/10.1177/0954406220941545.
25. Leshchenko, D., Ershkov, S., Kozachenko, T. (2022). Evolution of rotational motions of a nearly dynamically spherical rigid body with cavity containing a viscous fluid in a resistive medium International Journal of Non-Linear Mechanics. 142(3). 103980. https://doi.org/10.1016/j.ijnonlinmec.2022.103980.
26. Leshchenko, D., Ershkov, S., Kozachenko, T. (2023). Perturbed rotational motions of a nearly dynamically spherical rigid body with cavity containing a viscous fluid subject to constant body fixed torques. International Journal of Non-Linear Mechanics. 148(3). 104284. https://doi.org/10.1016/j.ijnonlinmec.2022.104284
27. Farag, A. M., Amer, T. S., Abady, I. M. (2022). Modeling and analyzing the dynamical motion of a rigid body with a spherical cavity. *Journal of Vibration Engineering and Technologies*. https://doi.org/10.1007/s42417-022-00470-7.
28. Routh, E. J. (2005). *Advanced Dynamics of a System of Rigid Bodies*. New York: Dover.
29. Bogoliubov, N. N., Mitropolsky, Yu. A. (1961). *Asymptotic Methods in the Theory of Nonlinear Oscillations*, Gordon and Breach Science, New York.

Література

1. Жуковский Н. Е. О движении твердого тела, имеющие полости, наполненные однородною капельною жидкостью. Избранные сочинения. Т. 1. М.-Л.: Гостехиздат, 1948. С. 31–152.
2. Moiseyev N. N., Rumyantsev V. V. Dynamic Stability of bodies Containing Fluid. New York: Springer, 1968. 345 p.
3. Lukovsky I. A. Mathematical Models for Rigid Bodies with a Liquid. Walter de Gruyter GmbH Co KG, 2015. 393 p.
4. Chernousko F. L. The Movement of a Rigid Body with Cavities Containing a Viscous Fluid. Washington: NASA, 1972. 214 p.
5. Chernousko F. L., Akulenko L. D., Leshchenko D. D. Evolution of Motions of a Rigid Body About its Center of Mass. Cham: Springer International Publishing, 2017. https://doi.org/10.1007/978-3-319-53928-7.
6. Kobrin A. I. On the motion of a hollow body with viscous liquid about its center of mass in a potential body force field. Journal of Applied Mathematics and Mechanics. 1969. № 33(3). pp. 418–427.
7. Smirnova E. P. Stabilization of free rotation of an asymmetric top with cavities completely filled with a fluid. Journal of Applied Mathematics and Mechanics. 1974. № 38(6). pp. 931–935.

8. Осипов В. З., Суликашвили Р. С. О колебании твердого тела со сферической полостью, цели-ком заполненной вязкой жидкостью, на эллиптической орбите. Тр. ин-та. Тбилис. мат.ин-та АН. Груз. ССР. 1978. Т. 58. С. 175–186.
9. Акуленко Л.Д., Лещенко Д.Д., Рачинская А.Л. Эволюция вращений спутника с полостью, заполненной вязкой жидкостью. Механика твердого тела. 2007. Вып. 37. С. 126–139.
10. Akulenko L. D., Zinkevich Y. S., Leshchenko D. D., Rachinskaya A. L. Rapid rotations of a satellite with a cavity filled with viscous fluid under the action of moments of gravity and light pressure forces. Cosmic Research. 2011. 49(5). 440–451. https://doi.org/10.1134/S0010952511050017.
11. Leshchenko D., Akulenko L., Rachinskaya A., Shchetinina Yu. Rotational motion of a satellite with viscous fluid under the action of the external resistance torque. Mathematics in Engineering, Science and Aerospace. 2015. № 6(3). pp. 383–391.
12. Baranova, E. U., Vil'ke, V. G. Evolution of motion of a rigid body with a fixed point and an ellipsoidal cavity filled with a viscous fluid. Moscow Univ. Mech. Bull. 2013. № 68(1). pp. 15–20.
13. Rachinskaya A. L. Motion of a solid body with cavity filled with viscous liquid. Cosmic Research. 2015. № 53(6). pp. 476–480.
14. Disser K., Galdi G.P., Mazzone G., Zunino P. Inertil motions of a rigid body with a cavity filled with a viscous liquid. Archive for Rational Mechanics and Analysis. 2016. № 221(1). pp. 487–526.
15. Ramodanov S. M., Sidorenko V. V. Dynamics of a rigid body with an ellipsoidal cavity filled with viscous fluid. International Journal of Non-Linear Mechanics. 2017. № 95. pp. 42–46. https://doi.org/10.1016/j.ijnonlinmec.2017.05.006.
16. Akulenko L. D., Leshchenko D. D., Rachinskaya A. L. Optimal deceleration of rotation of a dynamically symmetric body with a cavity filled with viscous liquid in a resistive medium. Journal of Computer and System Sciences International. 2010. № 49(2). pp. 222–226. https://doi.org/10.1134/S1064230710020073.
17. Akulenko L. D., Leshchenko D. D., Rachinskaya A. L. Optimal deceleration of rotation of an asymmetric body with a cavity filled with viscous fluid in a resistive medium. Journal of Computer and System Sciences International. 2012. № 51(1). pp. 38–48.
18. Neishtadt A. T. Evolution of rotation of a solid, acted upon by the sum of a constant and dissipative perturbing moments. Mechanics of Solids. 1980. № 15(6). pp. 21–27.
19. Пивоваров М. Л. О движении гироскопа с малым самовозбуждением. Известия АН СССР. Механика твердого тела. 1985. №6. С. 23–27.
20. Van der Ha J. C. Perturbation solution of attitude motion under body-fixed torques. Acta Astronautica. 1985. 12(10). pp. 861–869.
21. Kane T. R., Levinson D. A. Approximate description of attitude motion of a torque-free, nearly axisymmetric rigid body. Journal of the Astronautical Sciences. 1987. № 35(4). pp. 435–446.
22. Ayobi M. A., Longuski J. M. Analytical solution for translational motion of spinning-up rigid bodies subject to constant body-fixed forces and moments. Trans. ASME. Journal of Applied Mechanics. 2008. № 75(1). 011004/1-011004/8.
23. Romano M. Exast analytic solution for a rotation of a rigid body having spherical ellipsoid of inertia and subjected to a constant torque. Celestial Mechanics and Dynamical Astronomy. 2008. № 100. pp. 181–189.
24. Akulenko L. D., Leshchenko D. D., Paly K. S. Perturbed rotational motions of a spheroid with cavity filled with a viscous fluid. Proc. IMechE Part C: Journal of Mechanical Engineering Science. 2021. № 235(20). 4833–4837. https://doi.org/10.1177/0954406220941545.
25. Leshchenko D., Ershkov S., Kozachenko T. Evolution of rotational motions of a nearly dynamically spherical rigid body with cavity containing a viscous fluid in a resistive medium. International Journal of Non-Linear Mechanics. 2022. № 142(3). https://doi.org/10.1016/j.ijnonlinmec.2022.103980.
26. Leshchenko D., Ershkov S., Kozachenko T. Perturbed rotational motions of a nearly dynamically spherical rigid body with cavity containing a viscous fluid subject to constant body fixed torques. International Journal of Non-Linear Mechanics. 2023. № 148(3). 104284. https://doi.org/10.1016/j.ijnonlinmec.2022.104284.

27. Farag A. M., Amer T. S., Abady I. M. Modeling and analyzing the dynamical motion of a rigid body with a spherical cavity. Journal of Vibration Engineering and Technologies. 2022. https://doi.org/10.1007/s42417-022-00470-7.
28. Routh E. J. Advanced Dynamics of a System of Rigid Bodies. New York: Dover, 2005.
29. Bogoliubov N. N., Mitropolsky Yu. A. Asymptotic Methods in the Theory of Nonlinear Oscillations. New York: Gordon and Breach Science, 1961. 537 p.

Leshchenko Dmytro
Odessa State Academy of Civil Engineering and Architecture
DSc. in Physics and Mathematics, Professor
Didrikhson st. 4, Odessa, Ukraine, 65029
leshchenko_d@ukr.net,
ORCID: 0000-0003-2436-221X
Kozachenko Tetiana
Odessa State Academy of Civil Engineering and Architecture
Didrikhson st. 4, Odessa, Ukraine, 65029
PhD in Physics and Mathematics, Associate Professor
kushpil.t.a@gmail.com
ORCID: 0000-0001-9034-3776

For references:
Leshchenko D., Kozachenko T. (2022). Evolution of rotational motions in a resistive medium of a nearly dynamically spherical gyrostat subjected to constant body-fixed torques. Mechanics and Mathematical Methods. 4 (2). 19–31.

Для посилань:
Лещенко Д. Д., Козаченко Т. О. Еволюція обертальних рухів в середовищі з опором, близького до динамічно сферичного гіростата під дією постійних моментів в зв’язаних з тілом осях. Механіка та математичні методи, 2022. Т. 4. № 2. С. 19–31.